MATH 582 HOMEWORK 1
WEEK 1
Winter, 2009
Due January 23

Problem 1.

Exercise. *Let A, B and C be sets. Show the following.*
(a) $\big[A \subseteq C \ \wedge \ B \subseteq C\big] \ \rightarrow \ A \cup B \subseteq C$.
(b) $\big[C \subseteq A \ \wedge \ C \subseteq B\big] \ \rightarrow \ C \subseteq A \cap B$.

Proof. (a). Suppose $A \subseteq C$ and $B \subseteq C$. Fix any x with $x \in A \cup B$. Then, either $x \in A$ or $x \in B$. If $x \in A$ then $x \in C$ by the first assumptions; and if $x \in B$ then $x \in C$ by the second assumptions. Either way, $x \in C$.

(b). Suppose $C \subseteq A$ and $C \subseteq B$. Fix any $x \in C$. Then, $x \in A$ by the first assumption and $x \in B$ by the second assumptions, so $x \in A \cap B$. □

Problem 2.

Exercise. *Let A, B be sets. Show the following for any set C.*

$$A \subseteq B \ \leftrightarrow \ \textit{For any set } C, (C - B) \subseteq (C - A).$$

Proof. Fix sets A, B.
($\rightarrow$). Suppose $A \subseteq B$. Fix any set C and any x with $x \in C - B$. Then $x \in C$ and $x \notin B$. Since $A \subseteq B$, $x \notin A$ either. Thus, $x \in C - A$ by the definition of $-$.

($\leftarrow$). Suppose $(C - B) \subseteq (C - A)$ for any set C. Let $C = A$. Since $A - A = \emptyset$ we have $A - B = \emptyset$. Thus, there is no x with $x \in A$ and $x \notin B$. In other words, $A \subseteq B$. □

Problem 3.

Exercise. *Let A and B be sets. Show the following.*
(a) $B - (B - A) \subseteq A$.
(b) $A - B = A - (A \cap B)$.

Proof. (a). Fix any $x \in B - (B - A)$. So, $x \in B$ and $x \notin B - A$. From the latter, $x \notin B$ or $x \in A$. Since $x \in B$, we must have $x \in A$.

(b). Fix any x. Consider the following chain of equivalences.

$$\begin{aligned} x \in A - B &\leftrightarrow x \in A \wedge x \notin B \\ &\leftrightarrow x \in A \wedge x \notin A \cap B \\ &\leftrightarrow x \in A - (A \cap B). \end{aligned}$$

So, $\forall x\big[x \in A - B \leftrightarrow x \in A - (A \cap B)\big]$, and it follows by Extensionality that $A - B = A - (A \cap B)$.

□

Problem 4.

Exercise. *Let A and B be sets. Show the following.*

$$A \subseteq B \leftrightarrow A \subseteq B - (B - A)$$

Proof. Let A and B be sets.

($\rightarrow$). Suppose $A \subseteq B$. Fix any $x \in A$. Then, $x \in B$ by supposition and $x \notin B - A$ since $x \in A$. Thus, $x \in B - (B - A)$.

($\leftarrow$). Suppose $A \subseteq B - (B - A)$. Fix any $x \in A$. Then, $x \in B - (B - A)$, so $x \in B$.

□

Problem 5.

Exercise. *Determine the following.*
(a) $\cup\emptyset$.
(b) $\cap\emptyset$.

Proof. (a). $\cup\emptyset = \emptyset$. Since

$$x \in \cup\emptyset \leftrightarrow x \in y \text{ for some } y \in \emptyset,$$

but the right-side is impossible.

(b). $\cup\emptyset = V$, where V is the universal set (that is, $x \in V$ for all x). Since

$$x \notin \cup\emptyset \leftrightarrow x \notin y \text{ for some } y \in \emptyset,$$

but the right-side is impossible.

□