1 Introduction to Natural Deduction Proofs

Remark 9.1.1 Natural deduction proof systems constitute another type of proof mechanism, intended to formalize the kind of reasoning found in mathematical arguments. These are based on the idea of *subordinate proofs*, in which one introduces premises as assumptions and derives conclusions from these premises, then *discharges* those premises to produce assumption-free proofs. Unlike a refutation system, like semantic tableaux, natural deduction proofs attempt to prove tautologies directly. Failed natural deduction proofs to not produce counterexamples.

It is very common in mathematics to prove statements by making additional assumptions, which will later be discharged. Here are three examples of this type of reasoning.

1. To prove $\alpha \to \beta$: Assume α and show that β *must be true* by deriving β from this assumption. Of course, α may not be true, so the proper conclusion is that $\alpha \to \beta$ *must be true.*
2. To prove $\alpha \vee \beta$ it is rare that you can prove α or β directly. Instead: Assume $\neg\alpha$ and show that it *must be true* that β by deriving β from this assumption. Of course $\neg\alpha$ may not be true, in which case α is true, so the proper conclusion is that $\alpha \to \beta$ *must be true.*
3. To prove α when all else fails use *reductio ad absurdam*: Assume $\neg\alpha$ and show this leads to a contradiction. Then $\neg\alpha$ cannot be true, so α *must be true.*

All three examples make assumptions which may not be true, so the assumption must be discharged before a proper conclusion can be drawn. In each case, the added assumption may introduce information that is relevant for deriving some conclusion.

I will walk you through a simple example that combines the key elements of the proof method. The most natural rules have two sorts of rules for each connective. There is a rule for *introducing* the connective into the proof and a rule for *eliminating* the connective from the proof (by using it). This is an important principle in all systems of natural deduction proofs. A natural deduction proof is a sequence of propositions, so has no branching rules. However, natural deduction proofs do keep track of *active* assumptions and *discharged* assumptions to determine which earlier lines of the proof are accessible to later lines of the proof. Any instance of a proposition occurring under an active assumption is accessible, and no instance of a proposition occurring under a discharged assumption is accessible. Here are the natural rules for the conditional $\to$ which illustrate this.

The first rule is $\to$-elimination ($\to$E) and corresponds to the use of the *modus ponens* rule:

n	α	
$\vdots$	$\vdots$	
m	$\alpha \to \beta$	
$\vdots$	$\vdots$	
p	β	$\to$E, m, n

The rule says that if α and $\alpha \to \beta$ occur on preceeding lines (numbered n and m) of the proof that are *accessible* at the current line (numbered p), then you may extend the proof writing β on the next line. The comment to the right of line p explains that the p was derived from $\to$E from lines n and m. The order of occurrence of α and $\alpha \to \beta$ is irrelevant.

The second rule is the typical method of proving a conditional $\alpha \to \beta$: Assume α (a hypothetical assumption) and from this assumption derive β; once β has been derived, terminate the assumption and disallow further use of the assumption or anything derived from it (since α may very well be false). Here is the rule of $\to$-introduction ($\to$-I):

$$
\begin{array}{lll}
n & \vert\ \underline{\alpha} & \\
\vdots & \vert\ \vdots & \\
m & \vert\ \beta & \\
m+1 & \alpha \to \beta & \to\text{I}, n\text{–}m
\end{array}
$$

On line n we have introduced the hypothesis α and at the same time introduced a subproof by the vertical line on the left. This line runs continuously until it is terminated, and α and anything derived from α is accessible, until the subproof is terminated. Once β is derived on line m, the subproof is terminated at line $n + 1$ with the introduction of the proposition $\alpha \to \beta$. The continuous line to the left means that β lies to the immediate left of the subproof line introduced on line n. No proposition occurring in lines n to m are accessible after line $m + 1$. The comment to the right of $m + 1$ explains the inference was $\to$I based on the subproof running from lines n to m.

Here is an example of a proof that $(P \to Q) \to ((Q \to R) \to (P \to R))$. The goal is to prove this proposition follows from no assumptions.

$$
\begin{array}{lll}
1 & \vert\ \underline{P \to Q} & \\
2 & \vert\ \quad \vert\ \underline{Q \to R} & \\
3 & \vert\ \quad \vert\ \quad \vert\ \underline{P} & \\
4 & \vert\ \quad \vert\ \quad \vert\ Q & \to\text{E}, 1, 3 \\
5 & \vert\ \quad \vert\ \quad \vert\ R & \to\text{E}, 2, 5 \\
6 & \vert\ \quad \vert\ P \to R & \to\text{I}, 3\text{–}5 \\
7 & \vert\ (Q \to R) \to (P \to R) & \to\text{I}, 2\text{–}6 \\
8 & (P \to Q) \to ((Q \to R) \to (P \to R)) & \to\text{I}, 1\text{–}8
\end{array}
$$

At lines 4 and 5 any hypothesis was accessible, so we could use the hypotheses on lines 1 and 2. Once we discharged the hypothesis on line 3 at line 6, none of lines 3,4,5 were accessible in the proof. We must close subproofs in the reverse order they are introduced. For example, we could not terminate the subproof on line 2 until we have terminated the subproof on line 3. This makes sense since the propositions derived on lines 4 and 5 depended on the inner assumption on line 3. Since line 8 is the proposition we aimed to prove and it has been derived with no hypotheses left open. So, it is a tautology.

2 Rules for Natural Deduction Proofs

Remark 9.2.1 Natural deduction proofs first entered the literature with the work of Gerhard Gentzen in the early 30's. There are many different styles of proof, designed to capture the introduction and discharge of hypotheses, and many different rules for introducing and eliminating connectives. Our proof style here is due to William Fitch in the 50's. However, the rules we will use in our natural deduction proof system are based on the *uniform notation* from Lecture 7, and designed so that we can easily prove the Soundness and Completeness of the system. Uniform notation reduces every proposition to one of two types: those that behave *conjunctively* (type-A) and those that behave *disjunctively* (type-B). The rules of proof are quite natural to these types of behavior. However, these rules are only natural in the context of a propositional logic built from $\{\wedge, \vee, \neg\}$, so we will introduce *derived rules* for each connective in the next section which more naturally capture the way we reason using that connective.

Definition 9.2.2 (Active proposition) A proposition on a line is *active* on a later line in a proof if it does not occur behind an assumption which has been discharged. That is, if a proposition is introduced at a line in the proof it is active so long as the line to its immediate left continues through the proof. Once this line terminates, the proposition is no longer active.

Definition 9.2.3 (Rules for Natural Deduction)

The rules only apply to propositions which are still active on the line it is applied.

Reiteration. This is a trivial rule, but it is sometimes useful to repeat an earlier line later:

$$\begin{array}{lll} n & \alpha & \\ \vdots & \vdots & \\ m & \alpha & \text{R}, n \end{array}$$

Constants. The motivation for the constants is that we can always introduce $\top$ since it is always true, and if we ever derive $\perp$ we may conclude anything we like (since a contradiction implies everything).

$$\begin{array}{lll} n & \top & \top\text{I}, n \end{array} \qquad \begin{array}{lll} n & \perp & \\ \vdots & \vdots & \\ m & \alpha & \perp\text{E}, n \end{array} \qquad \begin{array}{lll} n & \alpha & \\ \vdots & \vdots & \\ m & \neg\alpha & \\ \vdots & \vdots & \\ p & \perp & \perp\text{I}, n, m \end{array}$$

Type-A. Each type-A proposition has an introduction and elimination rule. We assume that α is a type-A proposition whose components are α_1 and α_2. The rules clearly capture the conjunctive nature of type-A propositions.

$$\begin{array}{lll} n & \alpha & \\ \vdots & \vdots & \\ m & \alpha_1 & \text{AE}, n \end{array} \qquad \begin{array}{lll} n & \alpha & \\ \vdots & \vdots & \\ m & \alpha_2 & \text{AE}, n \end{array} \qquad \begin{array}{lll} n & \alpha_1 & \\ \vdots & \vdots & \\ m & \alpha_2 & \\ \vdots & \vdots & \\ p & \alpha & \text{AI}, n, m \end{array}$$

Type-B. Each type-B proposition has an introduction and elimination rule. We assume that β is a type-B proposition whose components are β_1 and β_2. The rules are less obvious because type-B propositions naturally branch into two possibilities, but our proofs are nonbranching – that is, our proofs only provide for what *must be true* given our current assumptions. I'll first state the rules, then explain them.

n	$\neg\beta_1$			n	$\neg\beta_2$		
$\vdots$	$\vdots$			$\vdots$	$\vdots$		
m	β			m	β		
$\vdots$	$\vdots$			$\vdots$	$\vdots$		
p	β_2	BE, m, n		p	β_1	BE, m, n	

Idea: if we know β holds and we have ruled out one of the components, then the other component *must be true*.

n	$\vert\ \neg\beta_1$ (assumption)			n	$\vert\ \neg\beta_2$ (assumption)	
$\vdots$	$\vert\ \vdots$			$\vdots$	$\vert\ \vdots$	
m	$\vert\ \beta_2$			m	$\vert\ \beta_1$	
$m+1$	β	BI, n–m		$m+1$	β	BI, n–m

Idea: If on the assumption that one of the components is false, we conclude the other component *must be true*, then we can conclude that β must be true.

Remark 9.2.4 The type-A rules are very natural rules for conjunctive reasoning, this is clear. The type-B rules are very natural rules for disjunctive reasoning. For example, the B-elimination rules capture the *disjunctive syllogism*:

1. It will rain or snow today.
2. The low temperature today is well above freezing, so snow is impossible.
3. Therefore, it will rain today.

The rule for B-introduction is also a common type of reasoning in mathematics. It is rare that we can prove a disjunction $\beta_1 \vee \beta_2$ by proving either β_1 must be true or β_2 must be true. More typical is the case where both are separately contingent, but jointly one must hold. By assuming one component is false, we have added information which may help us deduce the other component *must be true*.

Definition 9.2.5 A *proof* in a natural deduction system is a finite sequence of propositions $\delta_1, \delta_2, \ldots, \delta_n$ such that each proposition has either been introduced as an assumption or derived from earlier terms in the sequence by one of the inference rules in Definition 9.3.2.

A proposition δ is a *theorem* of a natural deduction system if δ is the last line of a natural deduction proof in which all assumptions have been discharged. We write $\vdash_{\mathrm{nd}} \delta$ if there is a natural deduction proof of δ.

Example 9.2.6 Lets show $\vdash_{\mathrm{nd}} P \to \neg\neg P$. This is a type-$B$ proposition whose components are $\beta_1 = \neg P$ and $\beta_2 = \neg\neg P$. The most natural guess how to start the proof is to try to use B-introduction, which gives us two possible strategies for starting the proof. Here is one proof where we assume $\neg\beta_1$.

1	$\vert\ \neg\neg P$ (assumption)	
2	$\vert\ \neg\neg P$	R, 1
3	$P \to \neg\neg P$	BI, 1–2

This proof also shows the usefulness of the reiteration rule. We could also prove the proposition starting with the assumption $\neg\beta_2$:

1	$\quad \neg\neg\neg P$	
2	$\quad \neg P$	AE, 1
3	$P \rightarrow \neg\neg P$	BI, 1–2

Here we used the fact that $\neg\neg\neg P$ is a type-A proposition whose component is $\alpha_1 = \neg P$ in the A-elimination rule.

Example 9.2.7 We show $\vdash_{\text{nd}} \neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q)$. We will use B-introduction where the components are $\beta_1 = \neg\neg(P \wedge Q)$ and $\beta_2 = (\neg P \vee \neg Q)$. It is not obvious which version of the B-introduction rule to use, we have chosen to start with $\neg \beta_2$.

1	$\quad \neg(\neg P \vee \neg Q)$	
2	$\quad \neg\neg P$	AE, 1
3	$\quad P$	AE, 2
4	$\quad \neg\neg Q$	AE, 1
5	$\quad Q$	AE, 4
6	$\quad P \wedge Q$	AI, 3, 5
7	$\quad \neg\neg(P \wedge Q)$	AI, 7
8	$\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q)$	BI, 1–7

It is also possible to start with the hypothesis $\neg \beta_1$. Since we now want to derive a type-B proposition β_2, we will need a second application of B-introduction:

1	$\quad \neg\neg\neg(P \wedge Q)$	
2	$\quad \neg(P \wedge Q)$	AE, 1
3	$\quad\quad \neg\neg P$	
4	$\quad\quad \neg Q$	BE, 2, 3
5	$\quad (\neg P \vee \neg Q)$	BI, 3–4
6	$\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q)$	BI, 1–5

The second derivation is shorter, but may be less clear since it more heavily depends on rules for type-B propositions.

Example 9.2.8 We now turn to reproving the proposition $\vdash_{\text{nd}} (P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$, but now using the rules given by the unified notation. This proposition is type-B and has components $\beta_1 = \neg(P \rightarrow Q)$ and $\beta_2 = ((Q \rightarrow R) \rightarrow (P \rightarrow R))$. It is not apparent which version of B-elimination to use, so we will start with the hypothesis $\neg \beta_1$.

1	$\quad \neg\neg(P \rightarrow Q)$	
2	$\quad P \rightarrow Q$	AE, 1

At this point we are going to have to have a look at $((Q \rightarrow R) \rightarrow (P \rightarrow R))$, which is a type-$B$ proposition whose components are $\beta_1 = \neg(Q \rightarrow R)$ and $\beta_2 = (P \rightarrow R)$. Since this our goal is to prove this proposition we again use B-introduction:

3	$\quad\quad \neg\neg(Q \to R)$	
4	$\quad\quad Q \to R$	AE, 3
5	$\quad\quad\quad\quad \neg\neg P$	
6	$\quad\quad\quad\quad P \to Q$	R, 2
7	$\quad\quad\quad\quad Q$	BE, 6, 5
8	$\quad\quad\quad\quad \neg\neg Q$	AI, 7
9	$\quad\quad\quad\quad R$	BE, 4, 8
10	$\quad\quad P \to R$	BI, 5–9
11	$(Q \to R) \to (P \to R)$	BI, 3–10
12	$(P \to Q) \to ((Q \to R) \to (P \to R))$	BI, 1–11

Hardly a “natural deduction” compared with the proof given earlier using the introduction and elimination rules natural to reasoning with conditionals. Note that the reiteration on line 6 was not strictly necessary, it is only a reminder that it was available for use.