1 Tautological Equivalence

We will study the relation of *tautological equivalence* in more detail.

Definition 5.1.1 Propositions α and β are *tautologically equivalent*, denoted by $\alpha \simeq \beta$, if both $\alpha \models \beta$ and $\beta \models \alpha$.

The relation $\simeq$ obeys some simple laws:

Lemma 5.1.2 $\simeq$ is an equivalence relation on **PROP**. That is, for all propositions α, β, γ,

$$\begin{aligned}&\alpha \simeq \alpha\\ &\alpha \simeq \beta \quad \text{implies} \quad \beta \simeq \alpha\\ &\alpha \simeq \beta \text{ and } \beta \simeq \gamma \quad \text{implies} \quad \alpha \simeq \gamma.\end{aligned}$$

If $\alpha \simeq \beta$ then these propositions *always* have the same truth value. For this reason there is a close conection between $\simeq$ and $\leftrightarrow$:

Lemma 5.1.3 The following are equivalent for all propositions α and β.

(a) $\alpha \simeq \beta$

(b) For all valuations v, $v(\alpha) = v(\beta)$.

(c) $\models \alpha \leftrightarrow \beta$.

Proof. $(a) \Rightarrow (b)$. If $\alpha \simeq \beta$ and v is any valuation such that $v(\alpha) = \mathbf{T}$, it must also be that $v(\beta) = \mathbf{T}$ since $\alpha \models \beta$. Similarly, any valuation such that $v(\beta) = \mathbf{T}$ it must also be that $v(\alpha) = \mathbf{T}$.

$(b) \Rightarrow (c)$. For any valuation v, you can verify from the truth table

$$v(\alpha \leftrightarrow \beta) = \mathbf{T} \quad \text{if and only if} \quad v(\alpha) = v(\beta).$$

So, if (b) holds for α and β, $v(\alpha \leftrightarrow \beta) = \mathbf{T}$ for every valuation v.

$(c) \Rightarrow (a)$. Suppose $\models (\alpha \leftrightarrow \beta)$, then $v(\alpha) = v(\beta)$ for every valuation v by the truth table conditions for $\leftrightarrow$. So, for any valuation v, if $v(\alpha) = \mathbf{T}$ then $v(\beta) = \mathbf{T}$ as well, which shows that $\alpha \models \beta$. Similarly, if $v(\beta) = \mathbf{T}$, then $v(\alpha) = \mathbf{T}$ as well, which shows that $\beta \models \alpha$. □

Remark 5.1.4 When $\alpha \simeq \beta$, then α is interchangeable with β in any proposition in which it occurs without changing the conditions under which the proposition is true. Intuitively this should be obvious by the previous lemma: α and β always have the same truth value.

Recall the definition of substitution.

Definition 5.1.5 (substitution operation) Let γ and δ be propositions of **PROP**. Then the operation of replacing all occurrences of δ by γ in a proposition α, denoted by α^δ_γ, is defined by structural recursion

$$\begin{aligned}
A^\delta_\gamma &= \begin{cases} \gamma & \text{if } \delta = A \\ A & \text{otherwise} \end{cases} \qquad \text{for any atom } A, \\
(\neg\,\alpha)^\delta_\gamma &= \begin{cases} \gamma & \text{if } \delta = (\neg\,\alpha) \\ \neg\,\alpha^\delta_\gamma & \text{otherwise,} \end{cases} \\
(\alpha \diamond \beta)^\delta_\gamma &= \begin{cases} \gamma & \text{if } \delta = (\alpha \diamond \beta) \\ \alpha^\delta_\gamma \diamond \beta^\delta_\gamma) & \text{otherwise} \end{cases} \\
& \qquad \text{for each } \diamond \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}
\end{aligned}$$

For example,

$$\big((((\neg\,Q) \vee P) \rightarrow (\neg\,Q))\big)^{(\neg\,Q)}_{(P\,\wedge\,R)} = (((P \wedge R) \vee P) \rightarrow (P \wedge R)).$$

We do need to show that α^δ_γ is a proposition.

Lemma 5.1.6 Let α, δ and γ be propositions. Then α^δ_γ is also a proposition.

Proof. The proof is by induction on α, where δ and γ remain fixed. Let S be the set of propositions α such that α^δ_γ is also a proposition. Clearly, if $\alpha^\delta_\gamma = \gamma$, then α^δ_γ is a proposition, so we will only consider those cases where this is not true.

If α is a propositional atom and $\delta \neq \alpha$, then $\alpha^\delta_\gamma = \alpha$, which is a proposition. So, the S contains all propositional atoms.

Suppose $\alpha, \beta \in S$, so that α^δ_γ and β^δ_γ are propositions. Then $(\neg\,\alpha)^\delta_\gamma = (\neg\,\alpha^\delta_\gamma)$, and $(\alpha \diamond \beta)^\delta_\gamma = (\alpha^\delta_\gamma \diamond \beta^\delta_\gamma)$ are propositions by the inductive hypothesis.

So, $S = \textbf{PROP}$; that is, α^δ_γ is a proposition for any α. □

Theorem 5.1.7 (Replacement Theorem) Let β and γ be a proposition and P a propositional symbol.

$$\beta \simeq \gamma \quad \Longrightarrow \quad \alpha^P_\beta \simeq \alpha^P_\gamma$$

Proof. The proof is by induction on α. If Q is a propositional symbol different from P (or one of $\bot$ or $\top$), then by the reflexivity of $\simeq$:

$$Q^P_\beta = Q \simeq Q = Q^P_\gamma.$$

If $Q = P$, then by hypothesis

$$P^P_\beta = \beta \simeq \gamma = P^P_\gamma.$$

Suppose ϕ and ψ are propositions and that $\phi^P_\beta \simeq \phi^P_\gamma$ and $\phi^P_\beta \simeq \phi^P_\gamma$ (inductive hypothesis).

Now let $\diamond \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. By the definition of substitution

$$(\phi \diamond \psi)^P_\beta = \phi^P_\beta \diamond \psi^P_\beta \ \text{ and } \ (\phi \diamond \psi)^P_\gamma = \phi^P_\gamma \diamond \psi^P_\gamma$$

By lemma 5.1.3, it is sufficient to show that for all valuations v,

$$v(\phi^P_\beta \diamond \psi^P_\beta) = v(\phi^P_\gamma \diamond \psi^P_\gamma).$$

Since

$$\begin{aligned}
\phi^P_\beta \simeq \phi^P_\gamma \ &\text{ implies that } \ v(\phi^P_\beta) = v(\phi^P_\gamma) \ \text{ and} \\
\psi^P_\beta \simeq \psi^P_\gamma \ &\text{ implies that } \ v(\psi^P_\beta) = v(\psi^P_\gamma)
\end{aligned}$$

by the definition of truth valuation

$$v(\phi^P_\beta \diamond \psi^P_\beta) = \mathcal{H}_\diamond\big(v(\phi^P_\beta, \psi^P_\beta\big) = \mathcal{H}_\diamond\big(v(\phi^P_\gamma, \psi^P_\gamma\big) = v(\phi^P_\gamma \diamond \psi^P_\gamma).$$

By the definition of substitution

$$(\neg\,\phi)^P_\beta = \neg\,\phi^P_\beta \text{ and } (\neg\,\phi)^P_\gamma = \neg\,\phi^P_\gamma$$

Similarly as above

$$v(\neg\,\phi^P_\beta) = \mathcal{H}_\neg\big(v(\phi^P_\beta)\big) = \mathcal{H}_\neg\big(v(\phi^P_\gamma)\big) = v(\neg\,\phi^P_\gamma).$$

This completes the inductive step. □

The next useful result shows the tautological equivalence is preserved under substitution. The proof is left as an exercise, but is similar to that of the previous lemma.

Lemma 5.1.8 (substitution lemma) Suppose $\alpha \simeq \beta$ and let P be a propositional symbol. Then for any proposition γ, $\alpha^P_\gamma \simeq \beta^P_\gamma$.

2 List of some tautological equivalences

Remark 5.2.1 We will also need to be able to *simultaneously substitute* for several propositions. We can define the simultaneous substitution

$$\alpha^{\delta_1\ldots\delta_k}_{\gamma_1\ldots\gamma_k}$$

by recursion, analogous to the case of single substitution in Definition 5.1.5, but now replacing δ_i with γ_i wherever it occurs in α. Analogous versions of the Replacement and Substitution Lemmas will also hold for these simultaneous substitutions.

We list some useful tautological equivalences. We have used propositional symbols P, Q, R in place of arbitrary propositions α, β, γ; however, by the substitution lemma 5.1.8 (generalized to simultaneous substitutions) we can replace the propositional symbols by arbitrary propositions. The equivalences can be shown using the truth table method.

1. Commutative laws:
   (a) $(P \wedge Q) \simeq (Q \wedge P)$
   (b) $(P \vee Q) \simeq (Q \vee P)$
   (c) $(P \leftrightarrow Q) \simeq (Q \leftrightarrow P)$.

   Note however that $(P \rightarrow Q) \not\simeq (Q \rightarrow P)$.
2. Associative Laws:
   (a) $((P \wedge Q) \wedge R) \simeq (P \wedge (Q \wedge R))$
   (b) $((P \vee Q) \vee R) \simeq (P \vee (Q \vee R))$
   (c) $(P \leftrightarrow Q) \leftrightarrow R) \simeq (P \leftrightarrow (Q \leftrightarrow R))$.

   Note however that $((P \rightarrow Q) \rightarrow R) \not\simeq (P \rightarrow (Q \rightarrow R))$.
3. Distributive Laws:
   (a) $(P \wedge (Q \vee R)) \simeq (P \wedge Q) \vee (P \wedge R))$
   (b) $(P \vee (Q \wedge R)) \simeq (P \vee Q) \wedge (P \vee R))$
   (c) $(P \rightarrow (Q \wedge R)) \simeq (P \rightarrow Q) \wedge (P \rightarrow R))$
   (d) $(P \vee Q) \rightarrow R) \simeq (P \rightarrow R) \vee (Q \rightarrow R))$

Note however that $(P \to (Q \vee R)) \not\simeq (P \to Q) \vee (P \to R))$ and $(P \wedge Q) \to R) \not\simeq (P \to Q) \wedge (P \to R))$

4. De Morgan's Laws:

   (a) $\neg(P \wedge Q) \simeq (\neg P \vee \neg Q)$

   (b) $\neg(P \vee Q) \simeq (\neg P \wedge \neg Q)$

5. Negation Laws

   (a) $\neg\neg P \simeq P$

   (b) $\neg(P \to Q) \simeq (P \wedge \neg Q)$

   (c) $\neg(P \leftrightarrow Q) \simeq (\neg P \leftrightarrow Q) \simeq (P \leftrightarrow \neg Q)$

6. Contraposition Laws

   (a) $(P \to Q) \simeq (\neg Q \to \neg P)$

   (b) $(P \leftrightarrow Q) \simeq (\neg P \leftrightarrow \neg Q)$