Remark 32.0.1 The goal in this lecture is to prove the soundness and completeness of the first-order form of the system of natural deduction

$$\Sigma \models \phi \quad \text{if and only if } \Sigma \vdash_{\text{nd}} \phi.$$

The argument here is an extension of that given in Lecture 16, Section 3 using the Model Completeness Theorem of Lecture 32.

1 Soundness of Natural Deduction

It will be useful to list some properties of logical consequence. These are easily shown from the definition of first-order truth. Conditions 4 and 6 were stated in Example 23.2.7 and proven in Homework 5.

Lemma 32.1.1

(a) (Weakening) If $\Sigma \models \alpha$ and $\Sigma \subseteq \Gamma$, then $\Gamma \models \alpha$.

(b) (Cut) If $\Sigma, \alpha \models \beta$ and $\Sigma \models \alpha$, then $\Sigma \models \beta$. More generally, if $\Sigma, \alpha_1, \ldots, \alpha_n \models \beta$ and $\Sigma \models \alpha_i$ for each $i \leq n$, then $\Sigma \models \beta$.

(c) (Universal Instantiation) $\Sigma, (\forall x)\phi \models \phi^x_t$ for any variable-free term t.

(d) (Universal Generalization) If $\Sigma \models \phi(a)$, then $\Sigma \models (\forall x)\phi^a_x$ provided x is substitutable for a in ϕ and a is a parameter that does not occur in Σ.

(e) (Existential Generalization) If $\Sigma \models \phi(t)$, then $\Sigma \models (\exists x)\phi^t_x$ provided x is substitutable for t in ϕ.

(f) (Existential Instantiation) If $\Sigma, \phi^x_a \models \psi$ and a does not occur in any of Σ, ϕ, ψ, then $\Sigma, (\exists x)\phi \models \psi$

Theorem 32.1.2 (Strong Soundness of Natural Deduction)

$$\Sigma \vdash_{\text{nd}} \phi \quad \text{implies} \quad \Sigma \models \phi$$

Proof. The proof of soundness will be by induction on the "length of proof". The quotes are because a partially completed natural deduction proof is no proof at all. Consider a proof of ϕ from premises Σ. At a given line of this proof we may have some assumptions $\gamma_1, \ldots, \gamma_k$ still active, not having been discharged, and a sentence Z on the last line. Although this is an incomplete proof of Z from Σ, it becomes a complete proof of Z from $\Sigma \cup \{\gamma_1, \ldots, \gamma_k\}$ if we replace these active assumption with premise introduction. (The assumption is no longer treated as hypothetical in the proof but is now taken from our set of initial premises.) Our incomplete proof now becomes an assertion:

$$\Sigma, \gamma_1, \ldots, \gamma_k \models Z$$

The strategy is to show that this assertion is always correct, at each line p of the proof of ϕ from Σ. On the last line all assumptions introduced during the proof will have been terminated so that we will have shown $\Sigma \vdash_{\text{nd}} \phi \quad \text{implies} \quad \Sigma \models \phi$.

The proof is by induction on the length p of a natural deduction proof. The first line of the proof must be a premise introduction from Σ or the introduction of a hypothetical assumption; in the first case $\Sigma \models \gamma$ when $\gamma \in \Sigma$ and in the second case $\Sigma, \gamma \models \gamma$. Suppose that at line p we have the sentence Z and the active assumptions $\gamma_1, \ldots, \gamma_n$. For the inductive hypothesis we suppose that for each $q < p$, if the sentence on line q is Y and the active assumptions are $\delta_1, \ldots, \delta_m$ then

$$\Sigma, \delta_1, \ldots, \delta_m \models Y.$$

The argument proceeds by considering the rule application on line p. The proof of soundness in the propositional case, Theorem 16.3.2, has shown that $\Sigma, \gamma_1, \ldots, \gamma_k \models Z$ when Z is deduced by an application of an introduction or elimination rule for a propositional connective. So, we consider on the quantifier rules here.

(Universal elimination). Suppose $Z = \psi^x_t$ and on an earlier line q we had $(\forall x)\psi(x)$. Each of the assumptions at line q are still active so included in $\gamma_1, \ldots, \gamma_n$. By the inductive hypothesis together with weakening (above) we have

$$\Sigma, \gamma_1, \ldots, \gamma_k \models (\forall x)\psi(x)$$

Applying universal instantiation (above) and cut (above) it follows that

$$\Sigma, \gamma_1, \ldots, \gamma_k \models \psi(t)$$

(Universal Introduction). Suppose $Z = (\forall x)\psi^a_x$ and on an earlier line q we had ψ, where x was substitutable for a in ψ. Each of the assumptions at line q are still active so included in $\gamma_1, \ldots, \gamma_n$. It also follows that a does not occur in any of the active assumptions $\gamma_1, \ldots, \gamma_n$ (by the hypothesis of the universal introduction rule) nor in Σ (since the parameters of the proof are new to Σ). It follows by the inductive hypothesis and weakening that

$$\Sigma, \gamma_1, \ldots, \gamma_k \models \psi.$$

Applying universal generalization (above) it follows that

$$\Sigma, \gamma_1, \ldots, \gamma_k \models (\forall x)\psi^a_x.$$

(Existential Introduction). Suppose $Z = (\exists x)\psi^t_x$ and on an earlier line q we had ψ, where x was substitutable for t in ψ. Each of the assumptions at line q are still active so included in $\gamma_1, \ldots, \gamma_n$. It follows by the inductive hypothesis and weakening that

$$\Sigma, \gamma_1, \ldots, \gamma_k \models \psi.$$

Applying existential generalization (above) it follows that

$$\Sigma, \gamma_1, \ldots, \gamma_k \models (\exists x)\phi^t_x.$$

(Existential Elimination). Suppose Z is the consequence of existential elimination from the assumption ψ^x_a on line q_1 and $(\exists x)\psi$ on line q_2. Each of the assumptions at line q_1 and q_2 are still active (except for the assumption made on line q_1) and so included in $\gamma_1, \ldots, \gamma_n$. It also follows that a does not occur in any of the active assumptions $\gamma_1, \ldots, \gamma_n$ (by the hypothesis of the existential elimination rule) nor in Σ (since the parameters of the proof are new to Σ). On line $p-1$ we deduced ψ using the hypothesis ψ^x_a, so by the inductive hypothesis and weakening it follows that each of the following are true:

$$\Sigma, \gamma_1, \ldots, \gamma_k \models (\exists x)\psi$$
$$\Sigma, \gamma_1, \ldots, \gamma_k, \psi^x_a \models Z$$

Applying existential instantiation (above) it follows that

$$\Sigma, \gamma_1, \ldots, \gamma_k, (\exists x)\psi \models Z$$

and so by cut

$$\Sigma, \gamma_1, \ldots, \gamma_k \models Z$$

□

2 Completeness of Natural Deduction

Definition 32.2.1 A set Σ is *natural deduction-consistent* (*ND-consistent*) if there is a proposition ϕ for which $\Gamma \nvdash_{\text{nd}} \beta$. Equvalently, $\Sigma \nvdash_{\text{nd}} \bot$. A set Γ is *natural deduction-inconsistent* if it is not natural deduction-consistent. That is, $\Gamma \vdash_{\text{nd}} \bot$.

Remark 32.2.2 The proof of completeness

$$\Sigma \vdash_{\text{nd}} \phi \quad \text{implies} \quad \Sigma \models \phi$$

is by contraposition: If $\Sigma \nvdash_{\text{nd}} \phi$, then $\Sigma \not\models \phi$. This is equivalent to showing the following:

$$\Sigma \text{ is ND-consistent} \quad \text{implies} \quad \Sigma \text{ is satisfiable.}$$

Since natural deduction is a proof system, it is more convenient to show that ND-inconsistency is a first-order inconsistency property. It then follows by Lemma 31.1.7 that ND-consistency is a first-order consistency property, and completeness follows by the Model Existence Theorem 31.2.1.

Here is a restatement of the definition of a first-order inconsistency property from Lecture 31for reference.

Definition 32.2.3 (First-Order Inconsistency Property) Let $\mathcal{L}$ be a first-order language and A be a set of parameters not in $\mathcal{L}$, and let $\mathcal{I}$ be a collection of sets of sentences in the extended language $\mathcal{L}^A$. We call $\mathcal{I}$ a *first-order inconsistency consistency property*, and the sets $S \in \mathcal{C}$ $\mathcal{I}$-inconsistent, if it meets the following conditions for each $S \in \mathcal{I}$:

(I1) Any set S with an atomic sentence and its negation, or $\bot \in S$ or $\neg \top \in S$ is $\mathcal{I}$-inconsistent.

(I2) If α is a type-A sentence with components α_1 and α_2 and $S \cup \{\alpha_1, \alpha_2\}$ is $\mathcal{I}$-inconsistent then so is $S \cup \{\alpha\}$.

(I3) If β is a type-B sentence with components β_1 and β_2, and both $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ are $\mathcal{I}$-inconsistent, then so is $S \cup \{\beta\}$.

(I4) If γ is a type-C sentence and $S \cup \{\gamma(t)\}$ is $\mathcal{I}$-inconsistent, where t is a term in the extended language, then $S \cup \{\gamma\}$ is $\mathcal{I}$-inconsistent .

(I5) If δ is a type-D sentence and $S \cup \{\delta(a)\}$ is $\mathcal{I}$-inconsistent, where a is a parameter not occurring in S or δ, then $S \cup \{\delta\}$ is $\mathcal{I}$-inconsistent

Remark 32.2.4 The conditions for first-order inconsistency properties are stated for type-C and type-D sentences, while the rules of natural deduction apply to universal and existential quantifiers. There needs to be a bridge lemma which shows that type-C sentences act universally according to natural deduction rules and type-D sentences act existentitally according to natural deduction rules. That is the point of the following lemma.

Lemma 32.2.5 The following hold for the natural deduction proof system.

1. $\vdash_{\text{nd}} \neg (\exists x)\phi \rightarrow (\forall x)\neg \phi$
2. $\vdash_{\text{nd}} \neg (\forall x)\phi \rightarrow (\exists x)\neg \phi$

Proof. For (1):

1	$\neg(\exists x)\phi$	
2	$\quad \phi^x_a$	
3	$\quad (\exists x)\phi$	$\exists$I, 2
4	$\quad \bot$	$\bot$I, 1, 3
5	$\neg\phi^x_a$	$\neg$I, 2–4
6	$(\forall x)\neg\phi$	$\forall$I, 5
7	$\neg(\exists x)\phi \rightarrow (\forall x)\neg\phi$	$\rightarrow$I, 1–6

For (2):

1	$\neg(\forall x)\phi$	
2	$\quad \neg(\exists x)\neg\phi$	
3	$\quad\quad \neg\phi^x_a$	
4	$\quad\quad (\exists x)\neg\phi$	$\exists$I, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\neg\phi^x_a$	$\neg$I, 3–5
7	$\quad \phi^x_a$	$\neg$E, 6
8	$\quad (\forall x)\phi$	$\forall$I, 7
9	$\quad \bot$	$\bot$I, 1, 8
10	$\neg\neg(\exists x)\neg\phi$	$\neg$I, 2–9
11	$(\exists x)\neg\phi$	$\neg$E, 10
12	$\neg(\forall x)\phi \rightarrow (\exists x)\neg\phi$	$\rightarrow$I, 1–11

□

Lemma 32.2.6 Natural deduction-inconsistency is a first-order inconsistency property. So, natural deduction-consistency is a first-order consistency property.

Proof. We showed that ND-inconsistency satisfies conditions (I1), (I2) and (I3) in Lemma 16.3.4 for propositional logic. That argument carries over to first-order logic, since there were no change in the rules for propositional connectives in first-order natural deduction.

(I4). Suppose γ is a type-C proposition and $S \cup \{\gamma(t)\}$ is ND-inconsistent with proof τ. If $\gamma = (\forall x)\phi$, then it using one application of universal elimination from $(\forall x)\phi$ to ϕ^x_t we can extend τ to a proof of a contradiction for $S \cup \{(\forall x)\phi\}$. If $\gamma = \neg(\exists x)\phi$, then starting with $S \cup \{\neg(\exists x)\phi\}$ use (1) to derive a proof of $\neg(\exists x)\phi \rightarrow (\forall x)\neg\phi$ and conditional elimination to get $(\forall x)\neg\phi$ and now use τ to get a proof of a contradiction as before.

(I5). Let a be a parameter with does not occur in Σ or δ and suppose $\Sigma \cup \{\delta(a)\}$ is ND-inconsistent with proof τ. If $\delta = (\exists x)\phi$, then we can start a proof of a contradiction with $(\exists x)\phi$ and then introduce ϕ^x_a as an assumption for existential elimination. τ provides the proof of a contradiction from the assumption of ϕ^x_a, so existential elimination provides a proof of a contradiction from $\Sigma \cup \{(\exists x)\delta\}$. If $\delta = \neg(\forall x)\phi$, then start with a proof of $\neg(\forall x)\phi \rightarrow (\exists x)\neg\phi$ as in (2) above and use $\neg(\forall x)\phi$ and conditional elimination to obtain $(\exists x)\neg\phi$. Now proceed to produce a contradiction using τ as before.

Now that ND-inconsistency is a first-order inconsistency property, it follows by Lemma 31.1.7 that ND-consistency is first-order consistency property.

□

Proof. Suppose $\Sigma \nvdash_{\text{nd}} \phi$. Then $\Sigma, \neg\phi \nvdash_{\text{nd}} \bot$: otherwise, we could convert a proof $\mathcal{P}$ of $\Sigma, \neg\phi \vdash_{\text{nd}} \bot$ to a proof of $\Sigma \vdash_{\text{nd}} \phi$ as follows:

1	$\quad\neg\phi$	
⋮	$\quad\mathcal{P}$	
n	$\quad\bot$	
$n+1$	$\neg\neg\phi$	$\neg$I, 1–n
$n+2$	ϕ	$\neg$E, $n+1$

So, $\Sigma \cup \{\neg\phi\}$ is natural deduction-consistent. □

Theorem 32.2.7 (Strong Completeness of Natural Deduction)

$$\Sigma \models \phi \quad \text{implies} \quad \Sigma \vdash_{\text{nd}} \phi$$

Proof. The proof is by contraposition. Suppose $\Sigma \nvdash_{\text{nd}} \phi$. Then $\Sigma, \neg\phi \nvdash_{\text{nd}} \bot$: otherwise, we could convert a proof $\mathcal{P}$ of $\Sigma, \neg\phi \vdash_{\text{nd}} \bot$ to a proof of $\Sigma \vdash_{\text{nd}} \phi$ as follows:

1	$\quad\neg\phi$	
⋮	$\quad\mathcal{P}$	
n	$\quad\bot$	
$n+1$	$\neg\neg\phi$	$\neg$I, 1–n
$n+2$	ϕ	$\neg$E, $n+1$

So, $\Sigma \cup \{\neg\phi\}$ is ND-consistent. Since ND-consistency is a first-order consistency property (by Lemma 32.2.6), it follow by the Model Existence Theorem that $\Sigma \cup \{\neg\phi\}$ is satisfiable. Thus, $\Sigma \not\models \phi$. □