1 Soundness Theorem

Remark 30.1.1 Our first goal is to prove the soundness of tableau proof. Recall that $\Sigma \vdash \alpha$ if there is a tableaux proof of α from premises Σ. Then the soundness theorem amounts to showing

$$\Sigma \vdash \alpha \quad \text{implies} \quad \Sigma \models \alpha$$

The approach is to prove the contrapositive: if there is a structure $\mathcal{A}$ satisfying $\Sigma \cup \{\neg\,\alpha\}$, then no tableau for Σ with root $\neg\,\alpha$ can be contradictory. The strategy is to argue that if π is some branch (not necessarily a completed path) in a tableau τ and π is satisfiable, in the sense that the sentences labeling the nodes of π are jointly satisfiable, then there is some extension of π in τ which is also satisfiable. This is not surprising given the close connection between the rules of construction for tableau and the definition of truth for structures as captured in the unified notation and Theorem 29.1.1.

Lemma 30.1.2 Let $\tau = \cup_n \tau_n$ is a tableau from a set Σ of sentences from a language $\mathcal{L}$. If $\mathcal{A}$ is any model for Σ (so all sentences of Σ are true in $\mathcal{A}$), then $\mathcal{A}$ can be expanded to an interpretation of the parameters U of the tableaux language so that all the sentences along some path π through τ is true in the expanded structure $\mathcal{A}^U$.

Proof. Let $\mathcal{A}$ be a structure for the language $\mathcal{L}$ in which all sentences of Σ are true in $\mathcal{A}$. Let $\tau = \cup_n \tau_n$ be any tableau for Σ. The sentences occurring in the tableau τ may have additional parameters from a set U of tableaux parameters which must be interpreted in $\mathcal{A}$. We generate both the path π and interpretation $c^{\mathcal{A}}$ for each parameter $c \in U$ by recursion on τ_n. At each step n, we will define a noncontradictory path π_n through τ_n and an extension $\mathcal{A}_n$ of $\mathcal{A}$ (with the same domain) which interprets all parameters c_i occurring in some sentence of τ_n. For $n = 0$, let $\mathcal{A}_0 = \mathcal{A}$ and $\pi_0 = \langle \phi \rangle$ be the root. Since $\phi \in \Sigma$, ϕ is true in $\mathcal{A}$.

Given τ_n, assume that π_n and $\mathcal{A}_n$ have been defined so that every sentence on π_n is true in $\mathcal{A}_n$. We will extend π_n to π_{n+1} and $\mathcal{A}_{n+1}$ to $\mathcal{A}_{n+1}$ to ensure that π_{n+1} is a path through τ_{n+1} extending π_n and every sentence of π_{n+1} is true in $\mathcal{A}_{n+1}$. If τ_{n+1} expands some path other than π_n, we do nothing, letting $\pi_n = \pi_{n+1}$ and $\mathcal{A}_n = \mathcal{A}_{n+1}$. If τ_{n+1} is obtained from τ_n by applying a rule to sentence X on π_n then we proceed by the type of the rule. If X is type-A or type-B, then we know that at one path extending π_n in τ_n is satisfiable in $\mathcal{A}_n$. So let $\mathcal{A}_{n+1} = \mathcal{A}_n$ and π_{n+1} be the extension of π_n in τ_{n+1}. The new cases handle type-C and type-C signed propositions.

$X = \gamma$. Then there is a ground term t such that $\gamma(t)$ and γ are added to π_n in τ_{n+1}. Note that if $a \in A$ is any element from the domain, then $\gamma(a)$ is true in $\mathcal{A}_n$ since $\gamma \leftrightarrow (\forall x)\gamma(x)$ is valid. (See Lecture 29, Section 1.) The term t may contain parameters not defined in $\mathcal{A}_n$, so fix *any extension* $\mathcal{A}_{n+1}$ of $\mathcal{A}_n$ which interprets these new parameters in the domain $\mathcal{A}$. Forexample, fix a in the domain of $\mathcal{A}$ and let $c^{\mathcal{A}_{n+1}} = a$ for each new parameter c in t. Extend π_n to π_{n+1} by adding $\gamma(t)$ and γ. It follows that each sentence of π_{n+1} is true in $\mathcal{A}_{n+1}$.

$X = \delta$. Then there is a parameter c new to π_n such that π_n is extended by adding $\delta(c)$ to τ_{n+1} and c has not yet occurred on π_n. This means that we have not yet fixed an interpretation of c in $\mathcal{A}_n$, so we are free to choose any interpretation. Since δ is true in $\mathcal{A}_n$, there is some element $a \in A$ from the domain of $\mathcal{A}$ so that $\delta(a)$ is true in $\mathcal{A}_n$. Extend $\mathcal{A}_n$ to $\mathcal{A}_{n+1}$ by letting $c^{\mathcal{A}_{n+1}} = a$ and extend π_n to π_{n+1} by adding $\delta(c)$ to π_n. So each sentence of π_{n+1} is true in $\mathcal{A}_{n+1}$. □

Theorem 30.1.3 (Completeness of the tableau method) If there is a tableau proof of α from Σ the $\Sigma \models \alpha$, i.e.

$$\Sigma \vdash \alpha \quad \text{implies} \quad \Sigma \models \alpha$$

Proof. If there is a structure $\mathcal{A}$ which satisfies $\Sigma \cup \{\neg\alpha\}$, then for any tableau for Σ starting with $\neg\alpha$, there is an extension $\mathcal{A}'$ of $\mathcal{A}$ to the new parameters in τ and path π through τ so that every sentence on π is true in $\mathcal{A}'$. So π is noncontradictory. Therefore, there can be no tableau proof of α from Σ. □

2 The Completeness Theorem

Remark 30.2.1 We now turn to the completeness theorem. As with the propositional proof, we will prove that any noncontradictory path π in a finished tableau τ is a Hintikka set, and so satisfiable by some structure by Hintikka's lemma 29.3.4. The goal in this section is to prove the Completeness Theorem for Semantic Tableaux. Later we will isolate the key properties used in the proof so that we can extend the theorem to other proof systems, such as natural deduction.

Theorem 30.2.2 If τ is a finished tableau and π a non-contradictory path through τ, then π is a Hintikka set. Thus, any non-contradictory finished path π is satisfiable.

Proof. Let π be non-contradictory finished path. Take the domain of π to be all ground terms in the language $\mathcal{L} \cup A$, where A are the parameters used in the tableau construction. Call this set of ground terms U.

There is no signed sentence and its conjugate on π, since π is noncontradictory, so (H0) is satisfied. Each of (H1), (H2), and (H4) are also satisfied since every node with a type-A, type-B or type-D sentence is reduced on a finished tableau. The condition (H3) is satisfied by condition (ii) of the definition of a finished tableau (26.1.3) which ensures that if γ is a type-C sentence on π, then each ground term $t \in U$ is eventually instantiated on π.

Since π is a Hintikka set, it follows by Hintikka's Lemma that π is satisfiable. □

Theorem 30.2.3 (Completeness theorem) If α is a logical consequence of Σ, then there is a tableau deduction of α from Σ, i.e.

$$\Sigma \models \alpha \quad \text{implies} \quad \Sigma \vdash \alpha$$

Proof. We prove the contraposition. Suppose $\Sigma \not\vdash \alpha$. It follows by Theorem 26.1.5 that there is a finished tableau τ from $\Sigma \cup \{\neg\alpha\}$ with root $\neg\alpha$. Then τ is noncontradictory by hypothesis. Let π be a noncontradictory path through τ. Then π is a Hintikka set, and so is satisfiable by Hintikka's lemma. Since each sentence in $\Sigma \cup \{\neg\alpha\}$ occurs along π, the set $\Sigma \cup \{\neg\alpha\}$ is satisfiable so that $\Sigma \not\models \alpha$.

□

3 Consequences of the Completeness Theorem

Remark 30.3.1 Interestingly, the next theorem was first stated and "proven" by Löwenheim in 1915. There were gaps in this proof, although the ideas at its core were quite inspired. The theorem, in its form below, was proven by Skolem in 1920 using a strong background assumption of the axiom of choice. In 1929 he improved the proof by eliminating the need for the axiom of choice.

The version of the theorem quoted here is called the *Downward Löwenheim-Skolem Theorem*, and continues to have many very unsettling ramifications to philosophically-minded logicians.

Theorem 30.3.2 (Löenheim-Skolem Theorem) **??** Let $\mathcal{L}$ be a countable language. Then every satisfiable set Σ of sentences of $\mathcal{L}$ is satisfiable in a countable structure.

Proof. Start a finished tableau τ from Σ with using parameters from a countable set $A = \{a_0, a_1, \ldots\}$. By the soundness theorem above there cannot be a contradictory tableau from Σ. So τ must have a noncontradictory path π. The ground terms occurring in τ all come from the set of constants of $\mathcal{L}$, which is countable, and the parameters A, which is countable. So, there are countably many ground terms occurring along π. The structure generated using Hintikka's Lemma uses the ground terms for its domain. So, there is a countable structure which satisfies Σ. □

Remark 30.3.3 Interestingly, the Löenheim-Skolem Theorem theorem was first stated and "proven" by Leopold Löwenheim in 1915, well before the Completeness Theorem for Predicate Logic. There were gaps in this proof, although the ideas at its core were quite inspired. The proof was completed by Skolem in 1920, introducing what are now called Skolem functions, but using a strong background assumption in the axiom of choice. In 1929 he improved the proof by eliminating the need for the axiom of choice.

The version of the theorem quoted here is called the *Downward Löwenheim-Skolem Theorem*, and had many very unsettling ramifications for the logicians at the time. For example, by 1930 logicians had recognized that all of the standard mathematics (analysis, geometry, abstract algebra, topology) at the time could be expressed and derived in a set of sentences expressed in first-order predicate language with one relation, set membership $\in$, a theory known as Zermelo-Frankel set theory. In this theory everything is a set (numbers, functions, spaces, algebras, etc.) But, since it is a first-order predicate language, if it is consistent (and most logicians believe it is), then there is a countable structure which is sufficient for doing all mathematics. To put this pictorially, we believe we live in a mathematical universe in which there are many sets which have too many elements to be counted-off using just the natural numbers. For example, there are far too many points on the real number line to enumerate them (see Cantor's diagonal construction in example **??**). However, but the Löenheim-Skolem Theorem we may well be living in a countable universe. This is not inconsistent with Cantor's result, since our mathematical world does not include the function enumerating our real numbers which would convince us that our universe is only countable. A countable mathematical universe very, very much smaller than we believe it is; perhaps, something like discovering our universe was but an atom among countless others in a very, very big universe.

There is an *Upward Löwenheim-Skolem Theorem*, which says that if a set of sentence from a countable language is satisfied in some infinite structure, it can be satisfied in structures of any infinite size. For example, there are more real numbers than natural numbers (which means that no matter how clever you are, you cannot produce an enumeration of the real numbers as $r_0, r_1, \ldots, r_n, \ldots$). The natural numbers can be axiomatized as a first-order set of sentences known as Peano Arithmetic. This language use $+, \cdot, =, 0$, but all results in this section carry over. So, by the Upward version of our theorem, there is a structure which satisfies all the axioms of Peano Arithmetic and has as many natural numbers as there are real numbers. Again, if we lived in a universe which was as rich with natural numbers as our real numbers, we would still lack the tools needed to recognize just how many our natural numbers we have.

Definition 30.3.4 (finitely satisfiable) A set of sentences Σ is *finitely satisfiable* if every *finite* subset of Σ is satisfiable.

Theorem 30.3.5 (Compactness Theorem for Predicate Logic) A set of propositions Σ is satisfiable if and only if it is finitely satisfiable.

Proof. Clearly if Σ is satisfiable, every subset is satisfiable. For the converse, suppose Σ is finitely satisfiable. We may suppose Σ is infinite. Suppose that Σ is unsatisfiable, so that $\Sigma \models \bot$. Then $\Sigma \vdash \bot$, so there is a proof τ of $\bot$ from Σ. Since every path through τ is finite (each is contradictory) and τ is finite branch (in fact, binary branching), it follows by König's Lemma that τ is finite. Let Σ_0 be the set of a sentences of Σ occurring at some node of τ. Since τ is finite, so is Σ_0. But Σ_0 is finite so satisfiable, and τ is a contradictory tableau from Σ_0. This contradicts the soundness theorem: $\Sigma_0 \vdash \bot$ but $\Sigma_0 \not\models \bot$. It follows that Σ must be satisfiable. □