Remark 27.0.1 Natural deduction proofs, unlike semantic tableaux, are based on inference steps which closely model deductive reasoning as it is carried out in mathematics. The rules for first-order logic extend those of propositional logic by adding 4 new rules for handling universal and existential quantifiers. Each connective has an *introduction rule* which provides a way to derive a sentence in which it is the main connective, and an *elimination rule* which provides a way to use a sentence in which it is the main connective in an inference.

1 Introduction

Let $\mathcal{A}$ be a structure with domain $A = \{a_0, a_1, \ldots$ and suppose the sentence $(\forall x)P(x)$ is true in $\mathcal{A}$. This means that each of the following sentences is true:

$$P(a_0) \quad P(a_1) \quad P(a_2) \quad \ldots$$

If $A = \{a_0, a_1, \ldots, a_n\}$ is finite we can actually express as a true conjunction:

$$P(a_0) \wedge P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_n)$$

Even if A were infinite, it is still true that the universal sentence $(\forall x)P(x)$ *behaves* (logically) as if it were asserting the conjunction of each instance. In fact the rules for working with universal statements are similar to the rules for conjunction. For example, here are the rules for conjunction elimination and universal elimination:

$$\begin{array}{lll} n & (\forall x)\phi(x) & \\ \vdots & \vdots & \\ m & \phi(t) & \forall\text{E}, n \end{array} \qquad\qquad \begin{array}{lll} n & \alpha \wedge \beta & \\ \vdots & \vdots & \\ m & \alpha & \wedge\text{E}, n \end{array}$$

Just as a conjunction asserts each conjunct, a universal sentence asserts each instance. This rule corresponds to a valid inference called *universal instantiation* (asserting an instance of a universal statement):

$$\Sigma, (\forall x)\phi(x) \models \phi(t) \qquad t \text{ is any ground term.}$$

So what about the introduction rule? The rule for conjunction is that you can assert a conjunction if you have already derived each conjunct:

$$\begin{array}{l|ll} n & \alpha & \\ \vdots & \vdots & \\ m & \beta & \\ \vdots & \vdots & \\ p & \alpha \wedge \beta & \wedge\text{I}, n, m \end{array}$$

If we were working in a finite domain $A = \{a_0, a_1, \ldots, a_n\}$ we could also conclude a universal statement $(\forall x)P(x)$ was true, once we confirmed that each instance $P(a_0), P(a_1), \ldots, P(a_n)$ was true. We can't do this in an infinite domain, or much less when we are trying to prove a universal sentence true, as in

$$(\forall x)\big((\forall y)R(x,y) \rightarrow (\exists y)R(x,y)\big).$$

Still, the rule for proving universal statements, an introduction rule, is not far off in spirit from the rule for conjunction of proving each instance. Here is how it works: if we want to prove a universal sentence

$(\forall x)P(x)$, we introduce a new name a and guarantee that we only assume facts about a that are *true of every element* in the domain. We then aim to prove $\phi(a)$. Suppose our proof proceeds as follows (once we introduce the name a):

$$\psi_1(a) \quad \psi_2(a) \quad \psi_3(a) \quad \ldots \psi_n(a) = \phi(a)$$

These statements may involve the name a. But, if we never assumed anything special about a, that was not true of every element in the domain, then we could just as well substitute an element of the domain b in place of a into the proof and justify the same sequence of derivations:

$$\psi_1(b) \quad \psi_2(b) \quad \psi_3(b) \quad \ldots \psi_n(b) = \phi(b).$$

So, the use of a provides a *proof template*: substitute the name of any element in the domain and the same argument will be valid for it. So, introducing a new name into the proof, in a way, is providing a proof of every instant. The inference from an "arbitrary instance" to every instance is known as *universal generalization*, and corresponds to the valid inference

$$\Sigma \models \phi(a) \quad \Rightarrow \quad \Sigma \models (\forall x)\phi(x) \qquad \text{provided } a \text{ does not occur in } \Sigma, \phi.$$

The proviso clues us in that we must be careful not to sneak "special assumptions" about a, and natural deduction proofs introduce a means of guarding against sneaking in such assumptions.

Suppose the sentence $(\exists x)P(x)$ is true in $\mathcal{A}$. This means that at least one of the following sentences is true:

$$P(a_0) \quad P(a_1) \quad P(a_2) \quad \ldots$$

If $A = \{a_0, a_1, \ldots, a_n\}$ is finite we can actually express as a true disjunction:

$$P(a_0) \vee P(a_1) \vee P(a_2) \wedge \ldots \vee P(a_n)$$

Even if A were infinite, it is still true that the existential sentence $(\exists x)P(x)$ *behaves* (logically) as if it were asserting the disjunction of each instance. Again, the rules for working with existential statements are similar to the rules for disjunction. For example, here are the rules for disjunction introduction and existential introduction:

n	$\phi(t)$			n	α	
$\vdots$	$\vdots$			$\vdots$	$\vdots$	
m	$(\exists x)\phi(x)$	$\exists$I, n		m	$\alpha \vee \beta$	$\vee$I, n

In the course of normal arguments, we almost never assert a disjunction $\alpha \vee \beta$ after we have established one disjunct (α say), because the disjunction merely logically waters down what we already new. Similarly, we almost never assert an existential statement $(\exists x)\phi(x)$ after a long and arduous proof that $\phi(t)$ for some term t, because we have again logically watered down what we have just proved. The rule for introducing an existential statement corresponds to a valid inference called *existential generalization*:

$$\Sigma, \phi(t) \models (\exists x)\phi(x) \qquad t \text{ is any ground term.}$$

We are arguing from an instance to a general statement (the existence of an element), so this is the existential counterpart of universal generalization.

So what about the elimination rule? In the case of disjunction the elimination rule was proof by cases: if I know $\alpha \vee \beta$ is true and I can show that γ follows from the assumption that α is true and it follows from the assumption that β is true, then γ is true. By deriving γ from each of α and β individually I am covering my bases. The same idea works for eliminating an existentially quantified statement. If we can deduce γ from each instance $\phi(t)$, then we can deduce γ from $(\exists x)\phi(x)$. Since we cannot expect to exhaust all possible instances $\phi(t)$, we have to provide a *proof template* that will work for any $\phi(t)$. We introduce a new name

a and assume that $\phi(a)$ together with only what we can prove for every element of the domain. If we are careful every statement we deduce using the parameter a:

$$\phi(a) = \psi_1(a) \quad \psi_2(a) \quad \psi_3(a) \quad \ldots \psi_n(a)$$

will hold of every element of the domain. So, if b names the actual witness, the sequence of inferences

$$\phi(a) = \psi_1(b) \quad \psi_2(b) \quad \psi_3(b) \quad \ldots \psi_n(b)$$

still holds with b replacing a. The inference from an "arbitrary instance" a to any instance is known *existential instantiation*, and corresponds to the valid inference

$$\Sigma\phi(a) \models \psi \quad \Rightarrow \quad \Sigma, (\exists x)\phi(x) \models \psi \qquad \text{provided } a \text{ does not occur in } \Sigma, \phi, \psi.$$

The proviso clues us in that we must be careful not to sneak "special assumptions" about a, and natural deduction proofs introduce a means of guarding against sneaking in such assumptions.

The existential and universal quantifier are not independent of each other, but are related to each other as conjunction is related by disjunction through the DeMorgan laws:

$$\models (\alpha \wedge \beta) \leftrightarrow \neg(\neg\alpha \vee \neg\beta) \qquad \models (\alpha \vee \beta) \leftrightarrow \neg(\neg\alpha \wedge \neg\beta)$$
$$\models (\forall x)\phi \leftrightarrow \neg(\exists x)\neg\phi \qquad \models (\exists x)\phi \leftrightarrow \neg(\forall x)\neg\phi$$

The treatment of the universal and existential quantifier as two sides of the "same coin" has been a source of discomfort to Logicians as well as students of Logic since the late nineteenth century development of logic. There is a school of logic, known as *Intuitionism*, which rejects this connection between the quantifiers. The real bone of contention is that an Intuitionist will only accept an existential statement $(\exists x)\phi$ if he can provide an actual instance $\phi(t)$ (or at least explain how to compute t!) A Classical Logician will accept $(\exists x)\phi$ if he can show that it is impossible for it to be false. This argument may fail to provide an actual instance. An example of this difference can be found in example 27.3.8

2 The Rules of Natural Deduction Proof

Remark 27.2.1 There are a number of subtle issues in the handling of quantifiers in our rules of natural deduction. Some of these issues we have faced in the rules of semantic tableaux, such as with the type-D rule, but others are new due to the nature of natural deduction proof with introduction rules to "build-up" sentences in the course of a proof.

Up to now we have only utilized substitution of a constant or parameter for a free variable. However, there is no reason why we cannot substitute one term for another in a sentence. One reason for this extension of substitution is provide for the existential introduction rule:

- Holmes trapped Moriarity.
- There someone trapped Moriarty.

We can symbolize this form of argument using $T(x, y)$ for x trapped y:

- $T(h, m)$.
- $(\exists x)T(x, m)$.

We have made the substitution of x for h in $T(h, m)$: $T(h, m)^h_x$.

Definition 27.2.2 (Substitution for a variable) Let u and v be any terms, then for any term t

$$t^u_v = \begin{cases} v & \text{if } t = u, \\ t & \text{otherwise.} \end{cases}$$

This is the result of substituting v for u in t.

For any terms u and v we define the substitution of v for u into a formula by recursion as follows: let ϕ and ψ are any formulae

$$
\begin{array}{rcl}
\big[Rt_1 \ldots t_n\big]^u_v & = & R([t_1]^u_v, \ldots [t_n]^u_v) \qquad \text{for each } n\text{-ary } R \in \mathbf{PS} \\
\big[\bot\big]^u_v = \bot & & \big[\top\big]^x_c = \top \\
\big[\neg\,\phi\big]^u_v & = & \neg\,\big[\phi\big]^u_v \\
\big[\neg\,\phi \diamond \psi\big]^u_v & = & \phi^u_v \diamond \psi^u_v \qquad \diamond \in \{\wedge\,,\,\vee\,,\,\rightarrow\,,\,\leftrightarrow\} \\
\big[(\forall y)\phi\big]^u_v & = & \begin{cases} (\forall y)\phi & \text{if } u = y \\ (\forall y)\big[\phi\big]^u_v & \text{otherwise.} \end{cases} \\
\big[(\exists y)\phi\big]^u_v & = & \begin{cases} (\exists y)\phi & \text{if } u = y \\ \exists y)\big[\phi\big]^u_v & \text{otherwise.} \end{cases}
\end{array}
$$

For example,

$$\big[(\forall y)(R(y,a) \;\rightarrow\; (\exists z)R(a,x)\big]^a_x = (\forall y)(R(y,a) \;\rightarrow\; (\exists z)R(x,z)$$

Remark 27.2.3 There is a potential difficulty with substituting a variable for another term, and this is when the variable becomes bound by a quantifier in some position. For example, consider the argument

- Someone trapped Moriarity.
- Someone was trapped.

(This seemed to be the best English expression, although it leaves it implicit that there was a trapper.) We can symbolize this form of argument using $T(x,y)$ for x trapped y:

- $(\exists y)T(y,m)$.
- $(\exists x)(\exists y)T(y,x)$.

where we used the substitution:

$$(\exists y)T(y,m)^m_x = (\exists y)T(y,x)$$

But suppose we used the variable y instead of x, then

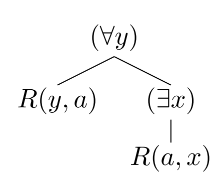
- $(\exists y)T(y,m)$.
- $(\exists y)(\exists y)T(y,y)$, which is equivalent to $(\exists y)T(y,y)$.

This is not a valid argument, because it concludes someone caught themselves, which may not be true. The problem is that in substituting y for m we placed y in a position where it was bound by the quantifier $(\exists x)$. This is called an *accidental binding* in the substitution. We need to be sure this does not happen when we introduce quantifiers.

We will say that a variable is *freely substitutable* for a term occurring at some location of the formula ϕ, if the variable does not become accidently bound when we substitute it for the parameter at that location. It is easier to see this by looking at a formation tree. Consider the sentence

$$\big[(\forall y)(R(y,a) \;\rightarrow\; (\exists x)R(a,x)\big]$$

whose formation tree is



The first occurrence of a is not beneath any node labeled $(\forall x)$ and $(\exists x)$, so no accidental binding will occur if x replaces a; but the second occurrence of a lies beneath a node labeled $(\exists x)$, so replacing a by x in this position would accidentally bind x. We say a variable may be *freely substituted* for an occurrence of a term t in a formula if x is not accidently bound when replacing t by x at that position.

Definition 27.2.4 (Freely substitutable) We define by induction the positions of a formula where a variable x is *freely substitutable* into a formula, to ensure no accidental bindings.

1. x is freely substitutable for any term in an atomic formulae.
2. x is freely substitutable into any position in $\neg\phi$ that it is freely substitutable in ϕ.
3. x is freely substitutable into any position in $\phi\diamond\psi$ that it is freely substitutable in ϕ and that it is freely substitutable in ψ. (Where $\diamond$ is any binary connective.)
4. When $x \neq y$, then x is freely substitutable into any position in $(\forall y)\phi$ or $(\exists y)\phi$ that it is freely substitutable in ϕ.
5. x is not freely substitutable in $(\forall x)\phi$ or $(\exists x)\phi$.

We say that x is *freely substitutable for a term t* in a formulae ϕ, if every x is freely substitutable into every position in which a occurs. For example, x is not freely substitutable for a in

$$(\forall y)(R(y,a) \rightarrow (\exists x)R(a,x).$$

Remark 27.2.5 (Universal and Existential Generalization) Existential generalization is the inference from a particular instance to an existential quantifier. For example, one valid inference which is an instance of generalization is

- Holmes has trapped Moriarty and Moriarty has trapped Watson.
- Therefore, Holmes trapped someone who has trapped Watson.

We could symbolize this in first-order logic as

- $T(h,m) \wedge T(m,w)$
- Therefore, $(\exists x)(T(h,x) \wedge T(x,w))$

Here we have replaced every occurrence of Moriarity by an existentially quantified variable. However, we could also infer

- Holmes has trapped Moriarty and Moriarty has trapped Watson.
- Therefore, Holmes has trapped Moriarty, while someone has trapped Watson.

symbolized as

- $T(h,m) \wedge T(m,w)$
- Therefore, $(\exists x)(T(h,m) \wedge T(x,w))$

Another valid inference is

- Holmes has trapped Moriarty and Moriarty has trapped Watson.
- Therefore, Holmes has trapped someone and Moriarty has trapped Watson.

symbolized as

- $T(h,m) \wedge T(m,w)$
- Therefore, $(\exists x)(T(h,x) \wedge T(h,w))$

So, in existential generalization one may replace a particular term in one or more places by an existentially quantified variable (if the variable may be freely substituted in those positions, of course).

In universal generalization, we replace a parameter, standing for an arbitrary member of the domain with a universally quantified variable. But the situation here is different than existential generalization since we cannot pick and choose whose positions we replace a variable. Consider the following argument:

- We say $x \leq y$ if there is some nonnegative z for which $x + z = y$. Let a be any number. Since $a + 0 = a$, it follows that $a \leq a$. So, $(\forall x) x \leq x$.

This is an instance of universal generalization:

- $a \leq a$
- Therefore, $(\forall x) x \leq x$

It is crucial that we make no special assumptions about, which is indeed the case in the argument. However, we may not pick and choose which occurrences of a we will generalize over. The following argument is not valid:

- $a \leq a$
- Therefore, $(\forall x) x \leq a$,
- And so, $(\exists x)(\forall y) x \leq y$.

The third line is not correct, since there is no largest number, but the inference from the second to third line is correct. The problem is the move from the first to second line. The problem is that every number bears a special relation to *themselves*, which they may not share with other numbers, so we cannot pick and choose which occurrences of the parameter a we replace – we must replace all occurrences.

This difference between universal and existential quantification will be reflected in our rules for natural deduction.

Definition 27.2.6 (Natural Deduction Rules for Quantifiers) Natural deductions proofs derive sentences in a first-order language $\mathcal{L}$. Natural deduction proof, like semantic tableaux, introduces an infinite set of private parameters $A = \{a_0, a_1, \ldots\}$ which are new to the language $\mathcal{L}$. The role of these parameters are different in natural deduction proofs, since we won't be using them to build a counterexample. In the rules that follow, a *ground term* is any term from the language $\mathcal{L}$ or a parameter from A.

Universal Quantifier. The rule for universal quantifier elimination, also called *Universal Instantiation*, is simply the elimination rule found in semantic tableaux:

n	$(\forall x)\phi(x)$	
$\vdots$	$\vdots$	
m	ϕ^x_t	$\forall$E, n

where t is any ground term. (The restriction to ground terms ensures that t is not a variable, so is freely substitutable into $\phi(x)$ for x.)

The rule for universal quantifier introduction, also called *Universal Generalization*, is trickier, because it uses parameters, but must ensure that there are no prior illegitimate assumptions about the parameter that is generalized:

$$\begin{array}{l|ll}
n & \phi(a) & \\
\vdots & \vdots & \\
m & (\forall x)\phi^a_x & \forall\text{I}, n\text{–}m
\end{array}$$

where ϕ^a_x is the result of replacing every occurrence of the parameter a occurring in ϕ by the variable x with two provisos:

1. x must be freely substitutable for every occurrence of a in ϕ,
2. a must not occur in any assumption which is accessible on line n where $\phi(a)$ occurs.

Existential Quantifier. The rule for existential quantifier introduction, also called *Existential Generalization* is straightforward

$$\begin{array}{lll}
n & \phi(t) & \\
\vdots & \vdots & \\
m & (\exists x)\phi(x) & \exists\text{I}, n
\end{array}$$

where t is any ground term, and $\phi(x)$ is the result of replacing zero or more occurrences of t by z and x is freely substitutable for t in each of these positions. (The case of substituting x for zero occurrences of t is legitimate, although pointless.)

The rule for universal quantifier elimination, also called *Existential Instantiation*, is similar to the rule used in semantic tableaux for type-D sentences. We do have to be sure that the parameter a has not been co-opted for another purpose. We could put a blanket restriction that the parameter *a not occur at all* in the proof. This is along the lines of the type-D rule in semantic tableaux. We liberalize the rule here so that it is closer to the liberalized type-D rule of Lecture 25.2.8:

$$\begin{array}{l|lll}
n & (\exists x)\phi & & \\
\vdots & \vdots & & \\
m & & \begin{array}{|l}\phi^x_a \\ \hline\end{array} & \\
\vdots & & \begin{array}{|l}\vdots\end{array} & \\
p & & \begin{array}{|l}\psi\end{array} & \forall\text{E}, n \\
m+1 & \psi & & \exists\text{E}, n, m\text{–}p
\end{array}$$

with two provisos:

1. a must not occur in any assumption which is accessible on line m when the assumption ϕ^x_a is introduced,
2. a must not occur in ψ.

I have also repeated the rules for the propositional connectives.

Definition 27.2.7 (Rules for Natural Deduction)

The rules only apply to propositions which are still active on the line it is applied. The rule of Reiteration and the rules for the constants remain the same.

Constants.

$$
\begin{array}{lll}
n & \top & \top\mathrm{I},\ n
\end{array}
\qquad
\begin{array}{lll}
n & \bot & \\
\vdots & \vdots & \\
m & \alpha & \bot\mathrm{E},\ n
\end{array}
\qquad
\begin{array}{lll}
n & \alpha & \\
\vdots & \vdots & \\
m & \neg\,\alpha & \\
\vdots & \vdots & \\
p & \bot & \bot\mathrm{I},\ n,\ m
\end{array}
$$

Negation Rules.

$$
\begin{array}{lll}
n & \neg\,\neg\,\alpha & \\
\vdots & \vdots & \\
m & \alpha & \neg\mathrm{E},\ n
\end{array}
\qquad
\begin{array}{l|ll}
n & \alpha & \\ \cline{2-2}
\vdots & \vdots & \\
m & \bot & \\
m+1 & \multicolumn{1}{l}{\neg\,\alpha} & \neg\mathrm{I},\ n\text{–}m
\end{array}
$$

Conjunction Rules.

$$
\begin{array}{lll}
n & \alpha \wedge \beta & \\
\vdots & \vdots & \\
m & \alpha & \wedge\mathrm{E},\ n
\end{array}
\qquad
\begin{array}{lll}
n & \alpha \wedge \beta & \\
\vdots & \vdots & \\
m & \beta & \wedge\mathrm{E},\ n
\end{array}
\qquad
\begin{array}{lll}
n & \alpha & \\
\vdots & \vdots & \\
m & \beta & \\
\vdots & \vdots & \\
p & \alpha \wedge \beta & A\mathrm{I},\ n,\ m
\end{array}
$$

Conditional Rules.

$$
\begin{array}{lll}
n & \alpha & \\
\vdots & \vdots & \\
m & \alpha \rightarrow \beta & \\
\vdots & \vdots & \\
p & \beta & \rightarrow\mathrm{E},\ m,\ n
\end{array}
\qquad
\begin{array}{l|ll}
n & \alpha & \\ \cline{2-2}
\vdots & \vdots & \\
m & \beta & \\
m+1 & \multicolumn{1}{l}{\alpha \rightarrow \beta} & \rightarrow\mathrm{I},\ n\text{–}m
\end{array}
$$

Disjunction Rules.

$$
\begin{array}{l|l|ll}
n & \multicolumn{3}{l}{\alpha \vee \beta} \\
m_1 & & \alpha & \\ \cline{3-3}
\vdots & & \vdots & \\
m_2 & & \gamma & \\
p_1 & & \beta & \\ \cline{3-3}
\vdots & & \vdots & \\
p_2 & & \gamma & \\
p_2+1 & \multicolumn{2}{l}{\gamma} & \vee\mathrm{E},\ n,\ m_1\text{–}m_2,\ p_1\text{–}p_2
\end{array}
\qquad
\begin{array}{lll}
n & \alpha & \\
\vdots & \vdots & \\
m & \alpha \vee \beta & \vee\mathrm{I},\ n
\end{array}
\qquad
\begin{array}{lll}
n & \beta & \\
\vdots & \vdots & \\
m & \alpha \vee \beta & \vee\mathrm{I},\ n
\end{array}
$$

Remark 27.2.8 We extend the definition of a proof to first-order sentence in the system of natural deduction. A natural deduction proof is defined for a first-order sentence in a language $\mathcal{L}$. A proof also uses an infinite set of parameters A, which are not part of the language $\mathcal{L}$. The role of these parameters is to make inferences with the introduction rule for the universal quantifier and the elimination rule for the existential quantifier.

Definition 27.2.9 A *proof* in a natural deduction system is a finite sequence of first-order sentences in the extended language $\mathcal{L}^A$, $\delta_1, \delta_2, \ldots, \delta_n$, such that each sentence has either been introduced as an assumption or derived from earlier terms in the sequence by one of the quantifier inference rules in Definition 27.2.6 or the propositional inference rules restated in Definition 27.2.7.

A sentence δ is a *theorem* of a natural deduction system if δ is the last line of a natural deduction proof in which all assumptions have been discharged. We write $\vdash_{\text{nd}} \delta$ if there is a natural deduction proof of δ.

We also extend natural deduction proof from a set of assumptions Σ from the language $\mathcal{L}$. Once again, none of the parameters from A occur in any of the assumptions of Σ. A *natural deduction proof of α from Σ* is a natural deduction proof of α in which sentences from Σ may be introduced at any point in the proof. We write $\Sigma \vdash_{\text{nd}} \alpha$ if there is a natural deduction proof of α from Σ.

3 Examples

Remark 27.3.1 (Parameters in existential elimination and universal introduction) The rules of existential elimination and universal introduction are special because of their introduction and use of parameters. In natural deduction, we introduce special hypotheses into the proof to further the inferences we make. These are hypothetical assumptions, which we introduce in a special subproof which governs the range in which the hypotheses occur. The inferences of existential instantiation (replacing an existential quantifier by a parameter) and universal generalization (replacing a parameter by a universal quantifier) depend crucially on our assumption that the parameters have no prior usage which would restrict their scope, because not applicable to every element of the domain. The restriction requiring the that the parameter not occur in an active assumption (or in the derived sentence ψ in the case of existential elimination) is reminiscent of the valid rules of inference:

(a) If $\Sigma, \phi(c) \models \psi$, then $\Sigma, (\exists x)\phi(x) \models \psi$, where c is a constant which does not occur in any of the sentences in Σ nor in $\phi(x), \psi$.

(b) If $\Sigma \models \phi(c)$ then $\Sigma \models (\forall x)\phi(x)$, where c is a constant which does not occur in any of the sentences in Σ nor in $\phi(x)$.

(see Homework 5).

There are only two ways a parameter can enter an argument in a natural deduction proof

(a) By the introduction of an assumption in $\rightarrow$-introduction, $\neg$-introduction $\vee$-elimination and $\exists$-elimination.

(b) By $\forall$-elimination.

Our restrictions allow that a parameter occurs in an accessible sentence, but just not in an active hypothesis. This means that any occurrence of the parameter in an accessible sentence was introduced by case 2, a $\forall$-elimination, so true for any term in place of the parameter.

Example 27.3.2 A derivation showing $\vdash_{\text{nd}} (\forall x)P(x) \rightarrow (\exists x)P(x)$.

1	$\quad (\forall x)P(x)$	
2	$\quad P(a)$	$\forall$E, 1
3	$\quad (\exists x)P(x)$	$\exists$I, 2
4	$(\forall x)P(x) \rightarrow (\exists x)P(x)$	$\rightarrow$I, 1–3

Just as in the semantic tableaux proof, we were forced to use a parameter in instantiate the universal quantifier on line 2. This is legitimate natural deduction proofs.

Example 27.3.3 The next derivation illustrates how to apply the rule of introduction for the universal quantifier. Here is a derivation showing $\vdash_{\text{nd}} (\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\forall x)P(x) \rightarrow (\forall x)Q(x))$.

1	$\quad (\forall x)(P(x) \rightarrow Q(x)$	
2	$\quad\quad (\forall x)P(x)$	
3	$\quad\quad P(a)$	$\forall$E, 2
4	$\quad\quad P(a) \rightarrow Q(a)$	$\forall$E, 1
5	$\quad\quad Q(a)$	$\rightarrow$E, 4, 3
6	$\quad\quad (\forall x)Q(x)$	$\forall$I, 3–5
7	$\quad ((\forall x)P(x) \rightarrow (\forall x)Q(x))$	$\rightarrow$I, 2–6
8	$(\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\forall x)P(x) \rightarrow (\forall x)Q(x))$	$\rightarrow$I, 1–7

Note that x is freely substitutable into $Q(a)$ and that the parameter a occurs in none of the hypotheses on lines 1 and 2.

Example 27.3.4 A derivation showing $\vdash_{\text{nd}} (\forall x)(\forall y)R(x,y) \rightarrow (\forall y)(\forall x)R(x,y)$.

1	$\quad (\forall x)(\forall y)R(x,y)$	
2	$\quad (\forall y)R(a,y)$	$\forall$I, 1
3	$\quad R(a,b)$	$\forall$I, 2
4	$\quad (\forall x)R(a,x)$	$\forall$E, 2–3
5	$\quad (\forall y)(\forall x)R(x,y)$	$\forall$E, 2–4
6	$(\forall x)(\forall y)R(x,y) \rightarrow (\forall y)(\forall x)R(x,y)$	$\rightarrow$I, 1–5

Example 27.3.5 A derivation showing $\vdash_{\text{nd}} (\forall x)R(x,y) \rightarrow (\exists x)(\exists y)R(x,y)$.

1	$\quad R(a,a)$	
2	$\quad (\exists y)R(a,y)$	$\exists$I, 1
3	$\quad (\exists x)(\exists y)R(x,y)$	$\exists$I, 2
4	$(\exists x)(\exists y)R(x,y)$	$\exists$E, 1–3

Line 2 of this example shows how we use the ability to pick and choose which positions of the term a we will substitute for.

Example 27.3.6 A derivation showing $\vdash_{\text{nd}} (\exists x)(\forall y)R(x,y) \rightarrow (\forall y)(\exists x)R(x,y)$.

1	$\quad (\exists x)(\forall y)R(x,y)$	
2	$\qquad (\forall y)R(a,y)$	
3	$\qquad R(a,b)$	$\forall$E, 2
4	$\qquad (\exists x)R(x,b)$	$\exists$I, 3
5	$\quad (\exists x)R(x,b)$	$\exists$E, 1, 2–3
6	$\quad (\forall y)(\exists x)R(x,y)$	$\forall$I, 2–5
7	$(\exists x)(\forall y)R(x,y) \rightarrow (\forall y)(\exists x)R(x,y)$	$\rightarrow$I, 1–6

On line 2 we introduced the parameter a to eliminate $(\exists x)$ and the parameter b on line 3 to eliminate the $(\forall y)$. On line 4 x was freely substitutable for a in $R(a,b)$. The existential elimination of line 5 was legitimate because $(\exists x)R(x,b)$ does not contain the parameter a. Finally, the introduction of the universal quantifier on line 6 was fine because y was freely substitutable for b in $(\exists x)R(x,b)$ and b has not appeared in any assumption. Note that we could not use the variable x to introduce $(\forall x)$ on line 6 since x is not free for b in $(\exists x)R(x,b)$.

Example 27.3.7 It is worth seeing an example of where a deduction can go wrong if a parameter appears in an assumption. The sentence $(\forall y)(P(y) \rightarrow (\forall x)P(x))$ is not valid. Consider the domain $\{1,2\}$ where $P^{\mathcal{A}} = \{1\}$. Here $P(1)$ is true in $\mathcal{A}$, but $(\forall x)P(x)$ is false, so the original sentence is false. The proof below violates one of our provisos in the universal introduction rule:

1	$\quad P(a)$	
2	$\quad (\forall x)P(x)$	incorrect application of $\forall$I, 1
3	$P(a) \rightarrow (\forall x)P(x)$	$\rightarrow$I, 1–2
4	$(\forall y)(P(y) \rightarrow (\forall x)P(x)$	incorrect application of $\forall$I, 3

Example 27.3.8 In this example we address another subtlety about the second proviso, of the universal introduction rule. You need to be careful about *which* instance of a sentence you are generalizing. In this example we show $\vdash_{\text{nd}} (\forall x)(P(x) \vee \neg P(x))$. The proof is very similar to the proof of $P \vee \neg P$ is propositional logic.

1	$\quad \neg(P(a) \vee \neg P(a))$	
2	$\qquad P(a)$	
3	$\qquad P(a) \vee \neg P(a)$	$\vee$I, 2
4	$\qquad \bot$	$\bot$I, 1, 3
5	$\quad \neg P(a)$	$\neg$I, 2–4
6	$\quad P(a) \vee \neg P(a)$	$\vee$I, 5
7	$\quad \bot$	$\bot$I, 1, 6
8	$\neg\neg(P(a) \vee \neg P(a))$	$\neg$I, 1–8
9	$P(a) \vee \neg P(a)$	$\neg$E, 9
10	$(\forall x)(P(a) \vee \neg P(a))$	$\forall$I, 10

The introduction of the universal quantifier $(\forall x)$ on line 10 is based on the instance of $P(a) \lor \neg P(a)$ on line 9, not the instance on line 1. The difference is that on line 9 we have proven that $P(a) \lor \neg P(a)$ is *valid* (it depends on no hypotheses), while on line 3 the instance $P(a) \lor \neg P(a)$ depends on the active assumption $P(a)$. It is important to keep this distinction in mind: at each point of a proof a line depends upon certain assumptions in play, it is the purpose of the side bars to make these assumptions explicit. It is unfortunately too common in actual arguments, where there is no such device, that assumptions are either implicitly invoked or are forgotten when the conclusion is state.

Example 27.3.9 In Example 25.2.8 we proved the following $(\exists y)\big((\exists x)P(x) \to P(y)\big)$, which motivated a liberalized version of the rule for type-D sentences, allowing the re-use of parameters, provided they were only introduced by type-C rule applications. The natural deduction proof of this fact is actually quite hard (I think) because it is so roundabout. Consider the difficulty

Line	Subproof depth	Formula
1	1 (assumption)	$(\exists x)P(x)$
2	2 (assumption)	$P(a)$
3	2	???

The problem is that we need to derive $(\exists x)P(x) \to P(a)$, but we cannot terminate the subproof for $(\exists x)P(x)$ until we release the guarded subproof for a, which we cannot do without losing $P(a)$. On the other hand, we cannot introduce a guarded subproof for a before we have assumed $(\exists x)P(x)$. There is no way out of this problem, except to try to do a proof by *reductio ad absurdam*:

Line	Subproof depth	Formula	Justification
1	1 (assumption)	$\neg(\exists y)\big((\exists x)P(x) \to P(y)\big)$	
2	2 (assumption)	$(\exists x)P(x)$	
3	3 (assumption)	$P(a)$	
4	4 (assumption)	$(\exists x)P(x)$	
5	4	$P(a)$	
6	3	$(\exists x)P(x) \to P(a)$	$\to$I, 4–5
7	3	$(\exists y)\big((\exists x)P(x) \to P(y)\big)$	$\exists$I, 6
8	3	$\bot$	$\bot$I, 1, 7
9	3	$P(b)$	$\bot$E, 8
10	2	$P(b)$	$\exists$E, 2, 3–9
11	1	$(\exists x)P(x) \to P(b)$	$\to$I, 2–10
12	1	$(\exists y)\big((\exists x)P(x) \to P(y)\big)$	$\exists$I, 11
13	1	$\bot$	$\bot$I, 1, 12
14	0	$\neg\neg(\exists y)\big((\exists x)P(x) \to P(y)\big)$	$\neg$I, 1–13
15	0	$(\exists y)\big((\exists x)P(x) \to P(y)\big)$	$\neg$E, 14

Rough, but there is no avoiding this. A weaker system of logic, known as intuitionistic logic, has the same quantifier rules and conditional rules, but $(\exists y)\big((\exists x)P(x) \to P(y)\big)$ is not valid for this system of logic. The difference is that intuitionistic logic does not recognize the use of double negation in line 15, so argument by *reductio ad absurdam* are not acceptable. This is a classic use *reductio ad absurdam* to prove *the existence* of some element in the domain, without actually *producing a witness*!! Think about what this sentence says: say $P(x)$ means "x is poor"

- There exists a person such that if anyone at all is poor, then this person is poor.

Consider how you might argue this sentence must be true: there is someone who is poor or no one is poor. If there is someone who is poor, this person witness to the fact that if anyone is poor, this person is. If no one is poor, then choose any person at all, so that if anyone is poor (which no one is), then this person is poor. Either way, there is such a person such that if anyone at all is poor, then this person is.

Here is another example of a paradoxical looking sentence: $(\exists y)\big(P(y) \rightarrow (\forall x)P(x)\big)$, which says

- There exists a person such that if this person is poor, then everyone is poor.

It is a first-order valid sentence, and you should try to prove it. Raymond Smullyan calls it the drinking sentence – where you can read $P(x)$ as "x drinks": there is a person who, if they take a drink, then everyone takes a drink. Strange, but true.

Example 27.3.10 Here is a derivation for $(\forall x)(P(x) \rightarrow Q(x)), (\exists x)P(x) \vdash_{\text{nd}} (\exists x)Q(x)$

1	$(\forall x)(P(x) \rightarrow Q(x))$	
2	$(\exists x)P(x)$	
3	$\quad$ $P(a)$	
4	$\quad$ $P(a) \rightarrow Q(a)$	$\forall$E, 1
5	$\quad$ $Q(a)$	$\rightarrow$E, 4, 3
6	$\quad$ $(\exists x)Q(x)$	$\exists$I, 5
7	$(\exists x)Q(x)$	$\exists$E, 3–6