1 Boolean Validity

Remark 24.1.1 Consider the following sentence:

$$(\forall x)P(x) \rightarrow ((\forall x)Q(x) \rightarrow ((\forall x)P(x) \wedge (\forall x)Q(x)))$$

This first-order sentence is valid. We don't need a sophisticated argument, since it is an "instance" of the tautology:

$$P \rightarrow (Q \rightarrow (P \wedge Q)),$$

where we have substituted $(\forall x)P(x)$ for P and $(\forall x)Q(x)$ for Q. The same reasoning that shows the proposition is a tautology also shows the first-order sentence is valid, since the truth conditions for propositional connectives were just taken over in giving the truth conditions of first-order sentences.

We fix a language $\mathcal{L}$ in what follows in this section.

Definition 24.1.2 (Boolean atom) By a *Boolean atom* we mean a sentence which is either an atomic sentence $P(c_1, \ldots, c_n)$ where P is an n-place predicate and $c_1, \ldots, c_n$ are constant symbols, or $\perp$ or $\top$ or a universal sentence (of the form $(\forall x)\phi$) or an existential sentence (of the form $(\exists x)\phi$).

Lemma 24.1.3 Every first-order formula is either a Boolean atom or built from Boolean atoms by applying the propositional connectives $\{\neg, \wedge, \vee, \rightarrow\}$.

Proof. The proof is by structural induction. Let X be the set of first-order formulae which are Boolean atoms or built from Boolean atoms by applying the propositional connectives $\{\neg, \wedge, \vee, \rightarrow\}$. If α is a atom, then it is a Boolean atom and so in X. If α and β are in X, then $(\forall x)\alpha$ and $(\exists x)\alpha$ are Boolean atoms and so in X. Each of $\neg\,\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$ and $\alpha \rightarrow \beta$ are also in X. □

Definition 24.1.4 (Boolean assignment, Boolean valuation) By a *Boolean assignment* we mean a function B from the Boolean atoms to the two truth values $\{\mathbf{T}, \mathbf{F}\}$.

A Boolean assignment is *arbitrary* from the point of view of a first-order valuation: A Boolean assignment is *any assignment* from Boolean atoms to truth values. For example, $(\forall x)(P(x) \wedge Q(x))$ and $(\forall x)P(x)$ are two distinct Boolean atoms, so a Boolean legitimate Boolean assignment is $B((\forall x)(P(x) \wedge Q(x)) = \mathbf{T}$ and $B((\forall x)P(x)) = \mathbf{F}$, although no first-order valuation could do this since

$$(\forall x)(P(x) \wedge Q(x)) \rightarrow (\forall x)P(x)$$

is valid as we showed in the previous section.

If B is a Boolean valuations, then the propositional valuation v_B given by the conditions Theorem 4.1.2 is a Boolean valuation.

The next theorem provides the obvious connection between Boolean valuations and first-order valuations.

Theorem 24.1.5 For any structure $\mathcal{A}$, let $B^{\mathcal{A}}$ be the Boolean assignment given by $B^{\mathcal{A}}(\alpha) = v_{\mathcal{A}}(\alpha)$ for any Boolean atom α. Let $v_{\mathcal{A}}$ be the first-order valuation generated from the structure $\mathcal{A}$ (given by Theorem 22.2.10) and $u_{B^{\mathcal{A}}}$ be the Boolean valuation extending $B^{\mathcal{A}}$ (given by Theorem 4.1.2). Then, $v_{\mathcal{A}}(\phi) = u_{B^{\mathcal{A}}}(\phi)$ for all first-order sentences.

Proof. Let $\mathcal{A}$ be a structure for a language $\mathcal{L}$, and consider the extension $\mathcal{L}^A$ of the language with constants from A. The proof is by induction on first-order sentences (in the extended sense of Theorem 21.3.4). Let X

be the set of sentences ϕ for which $v_{\mathcal{A}}(\phi) = u_{B^{\mathcal{A}}}(\phi)$. If ϕ is an atom, a universal sentence or an existential sentence, then it is a Boolean atom, so $v_{\mathcal{A}}(\phi) = u_{B^{\mathcal{A}}}(\phi)$. Suppose $v_{\mathcal{A}}(\phi) = u_{B^{\mathcal{A}}}(\phi)$. and $v_{\mathcal{A}}(\psi) = u_{B^{\mathcal{A}}}(\psi)$. Since $v_{\mathcal{A}}$ and $u_{\mathcal{A}}$ agree on the truth conditions for Boolean connectives, each of $v_{\mathcal{A}}(\neg\phi) = u_{B^{\mathcal{A}}}(\neg\phi)$ and $v_{\mathcal{A}}(\phi \diamond \psi) = u_{B^{\mathcal{A}}}(\phi \diamond \psi)$ for $\diamond \in \{\wedge, \vee, \rightarrow\}$. □

Definition 24.1.6 A first-order sentence ϕ is a *tautology* if it is true in every Boolean valuation. Similarly, a sentence α is a *tautological consequence* of a set of sentences Σ if every Boolean valuation which satisfies Σ also satisfies α.

Corollary 24.1.7 If ϕ is a tautology, then it is valid. If Σ tautologically implies α then $\Sigma \models \alpha$, that is Σ logically implies α.

Remark 24.1.8 The last corollary shows the relationship between logical implication and tautological implication, and allows us to take over many properties of tautological implication to logical implication:

1. If $\Sigma \models \alpha$ and $\Sigma \subseteq \Sigma^*$, then $\Sigma^* \models \alpha$.
2. $\Sigma, \alpha \models \beta$ if and only if $\Sigma \models \alpha \rightarrow \beta$, where α is a first-order sentence.
3. If $\Sigma, \alpha \models \beta$ and $\Sigma \models \alpha$, then $\Sigma \models \beta$.
4. $\Sigma \models \alpha$ and $\Sigma \models \beta$ if and only if $\Sigma \models \alpha \wedge \beta$.

2 Logical Equivalence

Remark 24.2.1 Recall that we are treating the biconditional as a defined connective:

$$\alpha \leftrightarrow \beta \quad \text{means} \quad (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha).$$

You can verify the following correctly provides the truth conditions for the biconditional in any structure $\mathcal{A}$ for the sentences α and β:

$$v_{\mathcal{A}}(\alpha \leftrightarrow \beta) = \begin{cases} \mathbf{T} & \text{if } v_{\mathcal{A}}(\alpha) = v_{\mathcal{A}}(\beta) \\ \mathbf{F} & \text{if } v_{\mathcal{A}}(\alpha) \neq v_{\mathcal{A}}(\beta) \end{cases}$$

Definition 24.2.2 We say that first-order formulae ϕ and ψ are *logically equivalent*, denoted by $\phi \simeq \psi$ if $\models \phi \leftrightarrow \psi$. If the universal closure of $\phi \leftrightarrow \psi$ is $(\forall x_1)\ldots(\forall x_n)(\phi \leftrightarrow \psi)$, then

$$\models \phi \leftrightarrow \psi \quad \text{if and only if} \models (\forall x_1)\ldots(\forall x_n)(\phi \leftrightarrow \psi).$$

The proof of the next two lemmas are simple consequences of the relation $\models$.

Lemma 24.2.3 The relation of logical equivalence is reflexive, symmetric and transitive: for every first-order formulae α, β, γ:

- $\alpha \simeq \alpha$,
- $\alpha \simeq \beta$ implies $\beta \simeq \alpha$,
- $\alpha \simeq \beta$ and $\beta \simeq \gamma$ implies $\alpha \simeq \gamma$

Lemma 24.2.4 Let α and β be formulae and let $x_1, \ldots, x_n$ be the variables occurring free in either α or β. Then the following are equivalent:

(a) $\alpha \simeq \beta$,

(b) $\models (\forall x_1)\ldots(\forall x_n)(\alpha \leftrightarrow \beta)$

(c) For any structure for α and β and any choice of elements $a_1, \ldots, a_n$ from the domain A,

$$v_{\mathcal{A}}(\alpha(a_1, \ldots, a_n) = v_{\mathcal{A}}(\beta(a_1, \ldots, a_n).$$

Note that if α and β are sentences, then condition (c) reduces to $v_{\mathcal{A}}(\alpha) = v_{\mathcal{A}}(\beta)$.

Remark 24.2.5 Let ϕ be any formula with its free variables among $x_1, \dots, x_n$, and let $c_1, \dots, c_n$ be constants new to the language. By theorem 23.2.6 we have

$$\models (\forall x_1) \dots (\forall x_n)\phi(x_1, \dots, x_n) \quad \text{if and only if} \quad \models \phi(c_1, \dots, c_n).$$

Be careful not to read too much into this!! It is NOT TRUE that

$$(\forall x_1) \dots (\forall x_n)\phi(x_1, \dots, x_n) \simeq \phi(c_1, \dots, c_n).$$

For example, let $P(x)$ be a 1-place predicate and c a constant, then $P(c) \to (\forall x)P(x)$ is not valid: let the domain be $\{1, 2\}$ and $P(1)$ true but $P(2)$ false. However, $P(c)$ is true in this structure when c is interpreted as 1.

However the following useful fact is true: let ϕ and ψ be formulae whose free variables are included in $x_1, \dots, x_n$ and $c_1, \dots, c_n$ new constants. Then

$$\phi(x_1, \dots, x_n) \simeq \psi(x_1, \dots, x_n) \quad \text{if and only if} \quad \phi(c_1, \dots, c_n) \simeq \psi(c_1, \dots, c_n)$$

by Theorem 23.2.6 when we apply the equivalent condition (b). In this case we have

$$\models (\forall x_1) \dots (\forall x_n)(\phi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$$

so that

$$\models \phi(c_1, \dots, c_n) \leftrightarrow \psi(c_1, \dots, c_n).$$

by theorem 23.2.6.

Remark 24.2.6 The next two example prove basic facts about quantifiers. Example 24.2.7 shows that the *order* of quantification does not matter and the second example 24.2.8 shows that "empty quantification" does not matter: $(\forall x)\phi$ and $(\exists x)\phi$ are equivalent to ϕ when x does not occur free in ϕ.

These examples also illustrate how extending a language with parameters can simplify arguments.

Example 24.2.7 The order of quantification does not matter: for any formula ϕ and distinct variables x and y,

$$(\forall x)(\forall y)\phi \simeq (\forall y)(\forall x)\phi \quad \text{and} \quad (\exists x)(\exists y)\phi \simeq (\exists y)(\exists x)\phi.$$

(If x and y are not distinct, then the formula on both sides are the same, so equivalent.)

In Homework 4 you will prove that when x and y are distinct variables:

$$[\phi^x_a]^y_b = [\phi^y_b]^x_b.$$

Let $x_1, \dots, x_n$ be the free variables in $(\forall x)(\forall y)\phi$, and $c_1, \dots, c_n$ new constants to the language. Using remark 24.2.5 above, it is sufficient to show

$$(\forall x)(\forall y)\phi(c_1, \dots, c_n) \simeq (\forall y)(\forall x)\phi(c_1, \dots, c_n) \quad \text{and} \quad (\exists x)(\exists y)\phi(c_1, \dots, c_n) \simeq (\exists y)(\exists x)\phi(c_1, \dots, c_n).$$

What this means is that we may assume $(\forall x)(\forall y)\phi$ is a sentence in our argument. Consider any structure $\mathcal{A}$ in which $(\forall x)(\forall y)\phi$ is true. Let a and b be any elements from the domain of A. So, $(\forall y)\phi^x_a$ is true in $\mathcal{A}$ by applying the truth condition for the universal quantifier, and again $[\phi(a_1, \dots, a_n)^x_a]^y_b$ is true in $\mathcal{A}$ by a second application. Since

$$[\phi^x_a]^y_b = [\phi^y_b]^x_a,$$

As a was arbitrary, the sentence $(\forall x)\phi^y_b$ is true in $\mathcal{A}$, and as b was arbitrary, the sentence $(\forall y)(\forall x)\phi$ is true in $\mathcal{A}$.

A similar argument shows that $(\exists x)(\exists y)\phi \simeq (\exists y)(\exists x)\phi$ holds as well.

Example 24.2.8 Let ϕ be any formula and x a variable that does not occur free in ϕ. Then

$$\phi \simeq (\forall x)\phi \quad \text{and} \quad \phi \simeq (\exists x)\phi.$$

It follows from remark 24.2.5 that we way assume that ϕ is a sentence (by replacing its free variables by new constants) in proving the equivalence. Let $\mathcal{A}$ be any structure for ϕ and suppose ϕ is true in $\mathcal{A}$. Let a be any element in the domain of A. Since x does not occur free in ϕ, $\phi^x_a = \phi$, so that ϕ^x_a is true as well. Since a was arbitrary, $(\forall x)\phi$ is true. Conversely, if $(\forall x)\phi$ is true, then $\phi = \phi^x_a$ is true (choosing any a in the domain).

As a consequence if $x_1, \ldots, x_n$ include all the free variables in ϕ (and possibly others as well, so that some of $x_1, \ldots, x_n$ may not occur free in ϕ), then $(\forall x_1) \ldots (\forall x_n)\phi$ is logically equivalent to the universal closure of ϕ.

The case for the existential quantifier is similar.

Theorem 24.2.9 For all first-order formulae $\alpha, \beta, \gamma, \delta$ the following properties hold: if $\alpha \simeq \beta$ and $\gamma \simeq \delta$ then

$$\begin{aligned} &(\alpha \diamond \gamma) \simeq (\beta \diamond \delta) \qquad \text{for } \diamond \in \{\wedge, \vee, \rightarrow\},\\ &(\neg\, \alpha) \simeq (\neg\, \beta),\\ &(\forall x)\alpha \simeq (\forall x)\beta,\\ &(\exists x)\alpha \simeq (\exists x)\beta \end{aligned}$$

Proof. The proof of the first two cases is similar to the proof of the Replacement theorem 5.1.7. Suppose $\alpha \simeq \beta$. It follows from remark 24.2.5 that we way assume that the only free variable (possibly) occurring in α and β is x (by replacing all other free variables by new constants). This means that the following holds

$$\models (\forall x)(\alpha(x) \leftrightarrow \beta(x)).$$

Let $\mathcal{A}$ be any structure in which $(\forall x)\alpha$ is true. Then for each $a \in A$, α^x_a is true and so β^x_a is true by (c) of Lemma 24.2.4. Since a was arbitrary, it follows that $(\forall x)\beta$ is true in $\mathcal{A}$. It similarly follows that $(\forall x)\alpha$ is true in any structure that $(\forall x)\beta$ is true.

Let $\mathcal{A}$ be any structure in which $(\exists x)\alpha$ is true. Then there is some $a \in A$ with α^x_a true and so β^x_a is true by (c) of Lemma 24.2.4. Thus, $(\exists x)\beta$ is true in $\mathcal{A}$. It similarly follows that $(\exists x)\alpha$ is true in any structure that $(\exists x)\beta$ is true. □

3 A Listing of Valid First-order Formulae

Let x and y be distinct variables, α and β formulae. Then

$$\begin{aligned} (\exists x)(\exists y)\alpha \;&\simeq\; (\exists y)(\exists x)\alpha \qquad & (\exists x)(\exists y)\alpha \;&\simeq\; (\exists y)(\exists x)\alpha\\ \neg\,(\exists x)\alpha \;&\simeq\; (\forall x)\neg\,\alpha & \neg\,(\forall x)\alpha \;&\simeq\; (\exists x)\neg\,\alpha\\ (\exists x)\alpha \;&\simeq\; \neg\,(\forall x)\neg\,\alpha & (\forall x)\alpha \;&\simeq\; \neg\,(\exists x)\neg\,\alpha\\ (\forall x)\alpha \wedge (\forall x)\beta \;&\simeq\; (\forall x)(\alpha \wedge \beta) & (\exists x)\alpha \vee (\exists x)\beta \;&\simeq\; (\exists x)(\alpha \vee \beta)\\ &\models (\forall x)\alpha \rightarrow (\exists x)\alpha\\ &\models (\exists x)(\forall y)\alpha \rightarrow (\forall y)(\exists x)\alpha. \end{aligned}$$

It is NOT generally true that you can switch the order of different quantifiers: $(\forall x)(\exists y)\alpha \rightarrow (\exists y)(\forall x)\alpha$ is not valid. Also be aware that while each of the following is true

$$(\exists x)(\alpha \wedge \beta) \rightarrow (\exists x)\alpha \wedge (\exists x)\beta \qquad (\forall x)\alpha \vee (\forall x)\beta \rightarrow (\forall x)(\alpha \vee \beta) \rightarrow$$

their converses are false.

Let α and β be any formulae and x a variable which does not occur free in γ. Then

$$
\begin{array}{rclcrcl}
(\forall x)\gamma & \simeq & \gamma & \quad & (\exists x)\gamma & \simeq & \gamma \\
\gamma \wedge (\forall x)\alpha & \simeq & (\forall x)(\gamma \wedge \alpha) & \quad & \gamma \vee (\forall x)\alpha & \simeq & (\forall x)(\gamma \vee \alpha) \\
\gamma \wedge (\exists x)\alpha & \simeq & (\exists x)(\gamma \wedge \alpha) & \quad & \gamma \vee (\exists x)\alpha & \simeq & (\exists x)(\gamma \vee \alpha) \\
\gamma \to (\forall x)\alpha & \simeq & (\forall x)(\gamma \to \alpha) & \quad & (\forall x)\alpha \to \gamma & \simeq & (\exists x)(\alpha \to \gamma) \\
\gamma \to (\exists x)\alpha & \simeq & (\exists x)(\gamma \to \alpha) & \quad & (\exists x)\alpha \to \gamma & \simeq & (\forall x)(\alpha \to \gamma)
\end{array}
$$