1 First-Order Validity

Remark 23.1.1 A sentence in a first-order language $\mathcal{L}$ is *valid* if it is true in every structure for the language. (That is, it is true under every interpretation.) However, unlike testing for being a tautology (which required checking only finitely many assignments to the propositional symbols), testing for validity requires checking truth in *infinitely many* structures. The difference is significant, and determining whether a sentence is valid or not is more involved.

Here are the truth conditions for the universal and existential quantifiers in a structure $\mathcal{A}$ for which we have names for the elements in its domain A:

$$v_{\mathcal{A}}((\exists x)\phi) = \begin{cases} \mathbf{T} & \text{if } v_{\mathcal{A}}(\phi^x_a) = \mathbf{T} \text{ for at least one } a \in A, \\ \mathbf{F} & \text{if } v_{\mathcal{A}}(\phi^x_a) = \mathbf{F} \text{ for all } a \in \mathcal{A}; \end{cases}$$

$$v_{\mathcal{A}}((\forall x)\phi) = \begin{cases} \mathbf{T} & \text{if } v_{\mathcal{A}}(\phi^x_a) = \mathbf{T} \text{ for all } a \in \mathcal{A}, \\ \mathbf{F} & \text{if } v_{\mathcal{A}}(\phi^x_a) = \mathbf{F} \text{ for at least one } a \in A; \end{cases}$$

Example 23.1.2 Show $(\forall x)P(x) \rightarrow (\exists x)P(x)$ is valid, where P is a 1-place predicate. The validity of this sentence depends crucially on the fact that domains are nonempty. Let $\mathcal{A}$ be any structure in which $(\forall x)P(x)$ is true. (In those structures in which $(\forall x)P(x)$ is false, the conditional is true anyhow.) Since the domain A is nonempty, there is some $a \in A$, so $P(a)$ is true in $\mathcal{A}$ by applying the truth conditions for the universal quantifier. So, $(\exists x)P(x)$ by applying the conditions for the existential quantifier in reverse.

Example 23.1.3 Show the following sentence is valid, where P, Q are 1-place predicates:

$$(\forall x)P(x) \wedge (\forall x)Q(x) \rightarrow (\forall x)(P(x) \wedge Q(x)).$$

Let $\mathcal{A}$ be any structure in which $(\forall x)P(x) \wedge (\forall x)Q(x)$ is true (the conditional is already true if its antecedent is false). To show $(\forall x)(P(x) \wedge Q(x))$ is true, we must show that for any $a \in A$, $P(a) \wedge Q(a)$ is true. Fix an arbitrary a in the domain A. Since $(\forall x)P(x)$ is true (a conjunction is true if and only if each conjunct is true), it follows that $P(a)$ is true (by the truth condition for a universal sentence), and since $(\forall x)Q(x)$ is true, it follows that $Q(a)$ is true as well. So, $P(a)$ and $Q(a)$ are true, and thus $P(a) \wedge Q(a)$ is true. Since a was arbitrary from the domain of $\mathcal{A}$, it follows that $(\forall x)(P(x) \wedge Q(x))$ is true in $\mathcal{A}$ by applying the condition for the universal quantifier in reverse.

The converse is also valid

$$(\forall x)(P(x) \wedge Q(x)) \rightarrow (\forall x)P(x) \wedge (\forall x)Q(x)$$

Suppose $(\forall x)(P(x) \wedge Q(x)$ is true in a structure $\mathcal{A}$. We will first show that $(\forall x)P(x)$ is true. Fix an arbitrary a in the domain A, so that $P(a) \wedge Q(a)$ is true by the truth conditions for a universal sentence. Thus, $P(a)$ is true. Since a was an arbitrary element from the domain A, $(\forall x)P(a)$ must be true as well. A parallel argument shows $(\forall x)Q(x)$ is also true. Thus, $(\forall x)P(x) \wedge (\forall x)Q(x)$.

Remark 23.1.4 Notice that an *argument* was required to establish validity. It is not possible to describe all possible structures interpreting the sentence. In fact the argument employs several devices which are hallmarks of quantifier reasoning. To prove the sentence is true in *every structure* (which provides an interpretation) we chose an *arbitrary structure* $\mathcal{A}$ and argued the sentence was true in the structure $\mathcal{A}$. If our goal was to prove a conditional was true, we assumed the antecedent true in the structure and argued the consequent must be true as well. You are well aware of hypothetical arguments in reasoning about the conditional from natural deduction proofs. Note that in structures in which the antecedent is false, the conditional is necessarily true, so we could safely ignore these cases. Next, in order to prove that the universal statement $(\forall x)P(x)$ was true, we had to show that

- For each $a \in A$, $P(a)$ is true in $\mathcal{A}$ (that is, $a \in P^{\mathcal{A}}$).

We chose an *arbitrary element* a from the domain A, assuming nothing else about this element. This allowed us to apply the true statement $(\forall x)(P(x) \wedge Q(x))$, which meant that

- For each $a \in A$, $P(a) \wedge Q(a)$ is true in $\mathcal{A}$ (that is, $a \in P^{\mathcal{A}}$ and $a \in Q^{\mathcal{A}}$.

The point is that it requires *deductive reasoning* to establish validity; it is not possible to construct a "truth table" which exhausts all possible cases. We will see that the method of proof by natural deduction captures these natural features of reasoning about generalities.

Example 23.1.5 Show that the following sentence is not valid:

$$((\exists x)P(x) \wedge (\exists x)Q(x)) \to (\exists x)(P(x) \wedge Q(x)).$$

We must produce a structure which is a counterexample: the antecedent $((\exists x)P(x) \wedge (\exists x)Q(x))$ is true but the consequent $(\exists x)(P(x) \wedge Q(x))$ is false. It requires some insight to see how to construct such a structure among the myriad of possibilities. The following structure $\mathcal{A}$ works:

- The domain is $A = \{1, 2\}$,
- The interpretations of the predicates are $P^{\mathcal{A}} = \{1\}$ and $Q^{\mathcal{A}} = \{2\}$.

So, $(\exists x)P(x)$ and $(\exists x)Q(x)$ are both true in $\mathcal{A}$, but there is no element $a \in A$ with $P(a)$ and $Q(a)$ both true in $\mathcal{A}$.

Remark 23.1.6 The method of semantic tableaux extended to first-order formulae provides a systematic method for produced counterexamples like the one from the previous example. There are sentences which require infinite domains to produce a counterexample. There is an example of this in Homework 4. The conjunction of the following three sentences provides one of the simplest examples (that I know of):

$$(\forall x)\neg R(x, x)$$
$$(\forall x)(\forall y)(\forall z)((R(x, y) \wedge R(x, z)) \to R(x, z))$$
$$(\forall x)(\exists y)R(x, y)$$

2 Extending Validity to Arbitrary First-Order Formulae

Definition 23.2.1 Let ϕ be a first-order formula and let $x_1, \ldots, x_n$ be distinct variables, then we will write $\phi(x_1, \ldots, x_n)$ to mean that all the free variables of ϕ are included in the list $x_1, \ldots, x_n$ (although not all the variables on the list need occur in ϕ). Let $\phi(a_1, \ldots, a_n)$ be the result of simultaneously substituting a_1 for x_1, a_2 for x_2 and so on.

Remark 23.2.2 We only defined truth on sentences, which are formulae without free variables. The point here is that if x occurs free in ϕ then the formula has no single fixed meaning in a structure $\mathcal{A}$. In this structure the formula $\phi(x_1, \ldots, x_n)$ represents the n-place relation on A

$$\big\{\langle a_1, \ldots, a_n\rangle : v_{\mathcal{A}}(\phi(a_1, \ldots, a_n)) = \mathbf{T}\big\}$$

We used this in constructing interpretations in Lecture 21, Section 1. The property of being a dog owner from Example 22.1.5 was the formula $(\exists y)(D(y) \wedge O(x, y))$, which has x free. The members of the domain for which this formula was true were dog owners.

Definition 23.2.3 Let ϕ be a formula of a language $\mathcal{L}$ whose free variables are $x_1, \ldots, x_n$. Then ϕ is *valid in a structure* $\mathcal{A}$ for $\mathcal{L}$ if the *universal closure* $(\forall x_1) \ldots (\forall x_n)\phi$ is true in $\mathcal{A}$. The formula ϕ is *valid* if it is valid in all structures for $\mathcal{L}$.

A set of formulae Σ with free variables for a language $\mathcal{L}$ is *satisfiable* if there is a structure in which all the formulae of Σ are valid (that is, their universal closures are true). Such a structure is called a *model* (or *interpretation*) for Σ.

Remark 23.2.4 For example, the formula $(\exists y)(D(y) \wedge O(x, y))$ is valid under an interpretation if its universal closure $(\forall x)(\exists y)(D(y) \wedge O(x, y))$ is true. Under the interpretation of Example 22.15, this would be the case if every member of the domain were a dog owner.

A convenient tool for proving validity (especially when this involves long strings of universal quantifiers, as occurs in the universal closures of formulae) is the technique of *extension by constants*. When we want to prove a universal sentence is true, such as $(\forall x)\phi$, we commonly begin "Let x be any arbitrary element from the domain ..." and proceed to prove ϕ. An notational alternative to this is to argue as follows: "Let a be an arbitrary element from the domain ..." and prove ϕ^x_a. It is crucial that we have not interpreted the name a previously, so that we freely take a to name any element in the domain. This type of reasoning, by introducing new constants, is valid, as we will now show in theorem 23.2.6 below. The method of extension by constants will also provide the basis of the treatment of quantifiers in semantic tableaux.

Definition 23.2.5 Let $\mathcal{L}$ be a language and $c_1, \ldots, c_n$ constants *not in the language*, which we will call *parameters* to $\mathcal{L}$, meaning that they have no intended interpretation in any structure for $\mathcal{L}$. Let $\mathcal{A}$ be any structure for $\mathcal{L}$, so that $c_1, \ldots, c_n$ are not interpreted in $\mathcal{A}$. Consider a sentence $\phi(c_1, \ldots, c_n)$ containing these new parameters. For elements $a_1, \ldots, a_n$ from the domain A, we will write $\phi(a_1, \ldots, a_n)$ to mean the simultaneous substitution of a_1 for c_1, ..., a_n for c_n into $\phi(c_1, \ldots, c_n)$. So, $\phi(c_1, \ldots, c_n)$ is true when we extend the structure $\mathcal{A}$ by interpreting c_1 as a_1, ..., c_n as a_n if and only if $\phi(a_1, \ldots, a_n)$ is true in $\mathcal{A}$.

We will say $\phi(c_1, \ldots, c_n)$ is *satisfiable in a structure* $\mathcal{A}$ if there is some $a_1, \ldots, a_n$ in the domain of $\mathcal{A}$ such that $\phi(a_1, \ldots, a_n)$ is true. We will say $\phi(c_1, \ldots, c_n)$ is *valid in a structure* $\mathcal{A}$ if $\phi(a_1, \ldots, a_n)$ is true for every $a_1, \ldots, a_n$ from the domain A. We will say $\phi(c_1, \ldots, c_n)$ is *satisfiable* if it is satisfiable in some structure, and *valid* if it is valid in every structure.

Theorem 23.2.6 Let $c_1, \ldots, c_n$ be parameters for a language $\mathcal{L}$ (that is, constants new to the language). Let $\phi(x_1, \ldots, x_n)$ be any formula from $\mathcal{L}$ (whose free variables are contained in the list $x_1, \ldots, x_n$). Then

1. $\phi(x_1, \ldots, x_n)$ is valid if and only if $\phi(c_1, \ldots, c_n)$ is valid
2. $(\exists x_1) \ldots (\exists x_n)\phi(x_1, \ldots, x_n)$ is satisfiable if and only if $\phi(c_1, \ldots, c_n)$ is satisfiable.

Note that $\phi(c_1, \ldots, c_n)$ is a *sentence* in the extended language obtained by substituting c_1 for x_1, ..., c_n for x_n into ϕ.

Proof. (1). Suppose ϕ is valid, which means that its universal closure $(\forall x_1) \ldots (\forall x_n)\phi$ is valid. Let $\mathcal{A}$ be any structure for $\mathcal{L}$. For any choice of $a_1, \ldots, a_n$ from the domain A, $\phi(a_1, \ldots, a_n)$ is true in $\mathcal{A}$, by applying the truth conditions for the universal quantifier n-times for each of $(\forall x_1), \ldots, (\forall x_n)$. Thus, $\phi(c_1, \ldots, c_n)$ is true for any choice of $a_1, \ldots, a_n$ with the interpretation c_1 by a_1, ..., c_n by a_n, and $\phi(c_1, \ldots, c_n)$ is valid in $\mathcal{A}$. Since the structure $\mathcal{A}$ was arbitrary, $\phi(c_1, \ldots, c_n)$ is valid.

The converse is the more interesting case. Suppose $\phi(c_1, \ldots, c_n)$ is valid. Let $\mathcal{A}$ be any structure for the language $\mathcal{L}$ and $a_1, \ldots, a_n$ elements from the domain $\mathcal{A}$. Then $\phi(a_1, \ldots, a_n)$ is true in $\mathcal{A}$, since $\phi(c_1, \ldots, c_n)$ is valid. We can now apply the truth conditions for the universal quantifier n times in reverse to conclude $(\forall x_1) \ldots (\forall x_n)\phi$ is true in $\mathcal{A}$. That is, since a_n was arbitrary, $(\forall x_n)\phi(a_1, \ldots, a_{n-1}, x_n)$ is true in $\mathcal{A}$, since a_{n-1} was arbitrary, $(\forall x_{n-1})(\forall x_n)\phi(a_1, \ldots, a_{n-2}, x_{n-1}, x_n)$ is true in $\mathcal{A}$, and so on.

(2). If $(\exists x_1) \ldots (\exists x_n)\phi(x_1, \ldots, x_n)$ is satisfiable, then there is a structure $\mathcal{A}$ in which $(\exists x_1) \ldots (\exists x_n)\phi(x_1, \ldots, x_n)$ is true. By applying the truth conditions for an existential quantifier n-times, there are elements $a_1, \ldots, a_n$ in the domain A in which $\phi(a_1, \ldots, a_n)$ is true. By interpreting c_1 as a_1, ..., c_n as a_n we have $\phi(c_1, \ldots, c_n)$ is satisfiable in $\mathcal{A}$.

Conversely, if $\phi(c_1, \ldots, c_n)$ is satisfiable, then there is a structure $\mathcal{A}$ and elements $a_1, \ldots, a_n$ with $\phi(a_1, \ldots, a_n)$ true in $\mathcal{A}$. By applying the truth conditions for the existential quantifier in reverse n-times, we have $(\exists x_1) \ldots (\exists x_n)\phi(x_1, \ldots, x_n)$ is true in $\mathcal{A}$, so satisfiable. $\square$

Example 23.2.7 Here are two other uses of extending the language with new constants to prove a generalization.

1. Suppose c is a constant which does not occur in any formula of Σ, nor in the formula $\phi(x)$ (where x is a variable which may be free in ϕ). If $\Sigma \models \phi(c)$, then $\Sigma \models (\forall x)\phi(x)$.

2. Suppose c is a constant which does not occur in any formula of Σ nor in ψ nor in the formula $\phi(x)$ If $\Sigma, \phi(c) \models \psi$, then $\Sigma, (\exists x)\phi(x) \models \psi$.

Often in mathematics we have proven some existential statement $(\exists x)\phi(x)$, but we don't know any witnesses for the existential statement, so we introduce a new name c and assume $\phi(c)$, and continue to argue. This reasoning is justified, provided the only assumption in play about c is that $\phi(c)$ is true. In (2) above, this key assumption is built into the proviso that c does not occur in any of our assumptions Σ nor in the statement we are trying to prove ψ nor in ϕ itself.

The proof of these examples will be later homework exercises.