1 Compactness Theorem

Remark 18.1.1 The applications of the compactness theorem is restricted to countable domains. This restriction is due to our assumption that the language of propositional logic has countably many propositional symbols. This restriction can be relaxed, although it requires stronger background theorems (such as Zorn's Lemma).

In this section we will use the Compactness Theorem as it was formulated in Homework 3.

Definition 18.1.2 A set Γ of propositions (possibly infinite) is *finitely satisfiable* if every finite subset Γ_0 is satisfiable.

Theorem 18.1.3 (Compactness Theorem) Every finitely satisfiable set of propositions is satisfiable.

This version is equivalent to the version proved in class:

$$\Gamma \models \beta \quad \text{implies that for some finite } \Gamma_0 \subseteq \Gamma,\, \Gamma_0 \models \beta.$$

It is convenient to introduce a general notation for strings of conjunctions and disjunctions.

Convention 18.1.4 If $\alpha_1, \ldots, \alpha_n$ are propositions we will write

$$\bigvee_{i=1}^{n} \alpha_i \quad \text{for} \quad \alpha_1 \vee \ldots \vee \alpha_n$$

and

$$\bigwedge_{i=1}^{n} \alpha_i \quad \text{for} \quad \alpha_1 \wedge \ldots \wedge \alpha_n$$

2 König's Lemma

Remark 18.2.1 We proved the compactness theorem for propositional logic using König's Lemma; however, we can also prove König's Lemma from the compactness theorem.

Theorem 18.2.2 Assume the compactness theorem 18.1.3. Then König's Lemma follows: every countably infinite finitely branching tree as an infinite path.

Proof. Let $\mathscr{T}$ be a countably infinite finitely branching tree. A node $\sigma \in \mathscr{T}$ is at level n if it has n ancestors. For example, the root is the only node at level 0, all children of the root are at level 1, all children of children are at level 2, and so on. We will write $\mathscr{T}_n$ for the nodes in the tree $\mathscr{T}$ at level n. If $\mathscr{T}$ is finitely branching, then every level has finitely many nodes. This can be proven by induction on the level n: for $n = 0$ there is only the root. Suppose there are finitely many nodes at level n (inductive hypothesis), say K nodes. The nodes at level $n+1$ are children of nodes at level n. Since $\mathscr{T}$ is finitely branching, each node at level n has finitely many children. Let M be a number larger than the number of children of any node at level n, then there are at most $K \cdot M$ nodes at level $n+1$.

Fix a set of propositional symbols which are indexed by the nodes of $\mathscr{T}$: $S = \{B_\sigma : \sigma \in \mathscr{T}\}$. (More formally, let $f : \mathscr{T} \to \mathbb{N}$ be an injection, then $B_\sigma = A_{f(\sigma)}$.) We will define a set of propositions which will guarantee the existence of an infinite path. If these propositions are satisfiable, then for any satisfying valuation v the set of true propositional symbols in S will constitute a path through $\mathscr{T}$:

$$\pi = \{\sigma : v(B_\sigma) = \mathbf{T}\}$$

will be an infinite path through $\mathscr{T}$.

These propositions come in three sorts:

(Γ_1) Let T_n be the nodes of $\mathscr{T}$ at level n. There are finitely many such nodes as we showed in the previous paragraph. The propositions in this group insist that some node of length n will be included in the path:

$$\Gamma_1 = \{ \bigvee_{\sigma \in T_n} B_\sigma \,:\, n \in \mathbb{N}\}.$$

These propositions say that there is a node of length n (generally, a node of depth n), so guarantee the path is infinite.

(Γ_2) This group of propositions ensure that if σ is on a path, then so is each each of its ancestors ρ (which we write as $\rho \subset \sigma$):

$$\Gamma_2 = \{(B_\rho \,\rightarrow\, B_\sigma) \,:\, \rho \subset \sigma, \rho, \sigma \in \mathscr{T}\}.$$

(Γ_3) The propositions of this group insist that there is no more than one node of length n (generally, a node of depth n) on any path:

$$\Gamma_3 = \{\neg\,(B_\sigma \,\wedge\, B_\rho) \,:\, \sigma, \rho \in \mathscr{T}_n, \sigma \neq \rho n \in \mathbb{N}\}.$$

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

We use the compactness theorem to show that Γ is satisfiable. Let Γ_0 be a finite subset of Γ. Let n be the largest number such that $\alpha = \bigvee_{\sigma \in \mathscr{T}_n} B_\sigma \in \Gamma_0$ (or let $n = 0$ if no such proposition occurs in Γ_0). Since $\mathscr{T}_n$ is non-empty, fix $\sigma \in \mathscr{T}_n$ and define a truth assignment v_σ by

$$v_\sigma(B_\rho) = \begin{cases} \mathbf{T} & \text{if } \rho \subset \sigma, \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

That is, $v_\sigma(B_\rho) = \mathbf{T}$ if and only if ρ is an ancestor of σ. You can verify that $v_\sigma(\alpha) = \mathbf{T}$ for each $\alpha \in \Gamma_0$. So, Γ is finitely satisfiable, and so is satisfiable by the compactness theorem.

Let $v : PS \rightarrow BOOL$ be a truth assignment which satisfies Γ. Let $\pi = \{\sigma \in \mathscr{T} \,:\, v(B_\sigma) = \mathbf{T}\}$. By Γ_1, π is infinite. By Γ_2 and Γ_3, π is actually a path through $\mathscr{T}$. □

3 Graph coloring

Definition 18.3.1 A *graph* consists is a pair $\langle V, E\rangle$ consisting of a set of vertices V and a relation $E \subseteq V \times V$. If uEv then we say that u is *adjacent* to v (or there is an *edge* from u to v.) We require that uEu is never true. A graph $G_1 = \langle V_1, E_1\rangle$ is a subgraph of G if $V_1 \subseteq V$ and $E_1 = E \cap V_1 \times V_1$.

A k-*coloring* of a graph $G = \langle V, E\rangle$ (where k is a positive integer) is a function $c : V \rightarrow [k] = \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever uEv. (That is, when there is an edge from u to v then u and v have different "colors", values of c.)

A graph G is said to be k-colorable if there exists a k-coloring of G.

Theorem 18.3.2 Let $G = (V, E)$ be a countable (possibly infinite) graph and let k be a positive integer. If every finite subgraph of G is k-colorable, then G is k-colorable.

Proof. Fix a set of propositional symbols $S = \{B_{u,i}\, u \in V, 1 \leq i \leq k\}$, where $A_{u,i}$ (when true) will be interpreted as "vertex u has the color i". There are three sets of propositions:

(Γ_1) This set will ensure that each vertex has some color:

$$\Gamma_1 = \{\bigvee_{i=1}^{k} B_{u,i} \, : \, u \in V\}.$$

(Γ_2) This set ensures that no vertex receives more than one color.

$$\Gamma_2 = \{\neg\,(B_{u,i} \wedge B_{u,j}) \, : \, u \in V, 1 \leq i, j \leq k, i \neq j\}.$$

(Γ_3) This set ensure that adjacent vertices receive different colors:

$$\Gamma_3 = \{\neg\,(B_{u,i} \wedge B_{v,i}) \, : \, u, v \in V, uEv, 1 \leq i \leq k\}.$$

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

We use the compactness theorem to show that Γ is satisfiable. Let Γ_0 be a finite subset of Γ. Let $V_0 = \{u_1, \ldots, u_n\}$ be all the vertices u such that $B_{u,i}$ occurs in some proposition in Γ_0. Let $G_0 = \langle V_0, E_0 \rangle$ be the subgraph of G determined by V_0. Fix a k-coloring $c : V_0 \to [k]$ (which exists by hypothesis). Define a a truth assignment $v : PS \to BOOL$ by

$$v(B_{u,i}) = \begin{cases} \mathbf{T} & \text{if } u \in V_0 \text{ and } c(u) = i, \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

You can verify that $v(\alpha) = \mathbf{T}$ for all $\alpha \in \Gamma_0$. So, Γ is finitely satisfiable, and is thus satisfiable by the compactness theorem.

Let $v : PS \to BOOL$ be an assignment satisfying Γ. Define a coloring $c : V \to [k]$ by $c(u) = i$ if and only if $v(B_{u,i}) = \mathbf{T}$. Since Γ_1 is satisfied every vertex v in V receives some color in $[k]$, and since Γ_2 is satisfied no vertex receives multiple colors (that is, c is really a function). Since Γ_3 is satisfied no adjacent vertices receive the same color. Therefore, G is colorable. □

Remark 18.3.3 The Four Coloring Theorem says that every planar graph is 4-colorable. By the preceding theorem, it suffices to prove that every *finite* planar graph is 4-colorable. The actual proof of the Four Coloring Theorem made something of a splash because it was first proven in 1976 by Kenneth Appel and Wolfgang Haken. They reduced the problem down to showing that there is a 4-coloring for 1936 finite planar graphs, then used a special-purpose computer program to check that each of these graphs was 4-colorable. Their proof has been considerably simplified, but to the best of my knowledge, the only proofs produced have been computer-assisted.