Remark 16.0.1 The method of semantic tableaux is a complete proof system because it was designed to establish its own completeness, in the sense that a *failed proof* leads to a construction of a Hintikka set formed from a finished non-contradictory path. More common among proof systems are those like the method of natural deduction, in which a failed proofs does not point in the direction of a counterexample, and may only mean a failure in our ingenuity.

The significance of Hintikka's Lemma 12.1.4 is that it draws a tight connection between syntax and semantics (the structure of propositions and their meanings). In this lecture we will abstract out the essence of the completeness proof for semantic tableaux in a form that allows us to extend completeness to our natural deduction proof system, as well as many other proof procedures for propositional logic. Furthermore, the abstraction we introduce here is readily extended to completeness proofs for first-order logic.

1 Propositional Consistency Property

Remark 16.1.1 (Completeness and consistency) Most completeness proofs argue by contraposition: if there is no proof of a proposition α, then there is a valuation v in which $v(\neg\beta) = \mathbf{T}$. It is useful to generalize this observation by introducing the notion of *consistency*.

- Let $\mathscr{P}$ be a proof system (such as semantic tableaux or natural deduction). A set of propositions is *$\mathscr{P}$-consistent* if no contradiction can be derived using the proof system's machinery.

In the case of tableaux, we could directly use its machinery to test whether $\Gamma \cup \{\neg\beta\}$ was tableaux-consistent: if not, we have a proof (a contradictory tableau), and if so, we have a Hintikka set (a noncontradictory finished path) satisfying all the propositions.

By looking carefully at such constructions, one can identify those features of consistency which are essential in ensuring a satisfying assignment exists. An *abstract consistency property* is a property which captures essential features of a notion of consistency and the *Model Existence Theorem* is the assertion that these features are sufficient to guarantee the existence of a suitable Boolean valuation. (The term *model* refers to the first-order counterpart of a Boolean valuation, a model or structure. The theorem we prove here will be extended to first-order Logic.)

Instead of talking about a consistency *property* of sets of propositions, we will talk about the *collection* $\mathcal{C}$ of all sets which are consistent (relative to whatever notion of consistency we are studying). We are identifying the collection of consistent sets of propositions with the property of being consistent. An abstract consistency property is defined to be a collection $\mathcal{C}$ of sets of propositions which meet certain closure conditions, very similar to the conditions used to identify Hintikka sets.

Definition 16.1.2 (Propositional Consistency Property) Let $\mathcal{C}$ be a collection of sets of propositions. We call $\mathcal{C}$ a *propositional consistency property*, and the sets $S \in \mathcal{C}$ $\mathcal{C}$-consistent, if it meets the following conditions for each $S \in \mathcal{C}$:

(C1) For any propositional symbol P, not both $P \in S$ and $\neg P \in S$. Also, $\bot \notin S$ and $\neg\top \notin S$.

(C2) If α is a type-A proposition with components α_1 and α_2 and $S \cup \{\alpha\}$ is $\mathcal{C}$-consistent, then so is $S \cup \{\alpha_1, \alpha_2\}$.

(C3) If β is a type-B proposition with components β_1 and β_2, and $S \cup \{\beta\}$ is $\mathcal{C}$-consistent, then *at least one of* $S \cup \{\beta_1\}$ or $S \cup \{\beta_1\}$ is $\mathcal{C}$-consistent.

Example 16.1.3 Let $\mathcal{C}$ be the collection of all satisfiable sets of propositions. Then $\mathcal{C}$ is a propositional consistency property:

(C1) No satisfiable set of propositions S can contain a propositional symbol and its negation nor $\bot$ nor $\neg\top$.

(C2) If $S \cup \{\alpha\}$ is satisfiable and $\alpha \in S$ is a type-A proposition with components α_1 and α_2, then $S \cup \{\alpha_1, \alpha_2\}$ is satisfiable, because $\alpha \simeq (\alpha_1 \wedge \alpha_2)$. (So, if $v(\alpha) = \mathbf{T}$, then $v(\alpha_1) = \mathbf{T}$ and $v(\alpha_2) = \mathbf{T}$.)

(C3) If S is satisfiable and $\beta \in S$ is a type-B proposition with components β_1 and β_2, then *at least one of* $S \cup \{\beta_1\}$ or $S \cup \{\beta_1\}$ is satisfiable, because $\beta \simeq (\beta_1 \vee \beta_2)$. (So, if $v(\beta) = \mathbf{T}$, at least one of $v(\beta_1) = \mathbf{T}$ or $v(\beta_2) = \mathbf{T}$.)

When working with proof systems it is usually more convenient to work with the contrapositive form:

Definition 16.1.4 (Propositional Inconsistency Property) Let $\mathcal{I}$ be a collection of sets of propositions. We call $\mathcal{I}$ a *propositional inconsistency property*, and the sets $S \in \mathcal{I}$ $\mathcal{I}$-inconsistent, if it meets the following conditions for each $S \in \mathcal{I}$:

(I1) Any set S with both $P \in S$ and $\neg P \in S$ for some propositional symbol P, or $\bot \in S$ or $\neg\top \in S$ is $\mathcal{I}$-inconsistent.

(I2) If α is a type-A proposition with components α_1 and α_2 and $S \cup \{\alpha_1, \alpha_2\}$ is $\mathcal{I}$-inconsistent then so is $S \cup \{\alpha\}$ is $\mathcal{I}$-inconsistent.

(I3) If β is a type-B proposition with components β_1 and β_2, and both $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ are $\mathcal{I}$-inconsistent, then so is $S \cup \{\beta\}$.

Example 16.1.5 The collection of *unsatisfiable* sets of propositions is a propositional inconsistency property, as you can verify. We will show that the collection of all tableau-inconsistent sets of propositions is a propositional inconsistency property. Recall that a set S is tableau-inconsistent if there is a contradictory tableau for S.

(C1) If S contains some propositional symbol P and its negation $\neg P$, then S is tableau-inconsistent. If S contains $\bot$ or $\neg\top$ then S is tableau-inconsistent.

(C2) Suppose $S \cup \{\alpha_1, \alpha_2\}$ is tableau inconsistent with contradictory tableau τ. This tableau may introduce α_1 and α_2 as part of the introduction rule from premises. The following will be a contradictory tableau for $S \cup \{\alpha\}$:

$$\begin{array}{c} \alpha \\ \tau' \end{array}$$

where τ' is obtained from τ by replacing all occurrences of α_1 and α_2 with the pair of α_1 followed by α_2, so that τ' is also contradictory. Each occurrence of α_1 and α_2 in τ' is justified by the type-A rule.

(C3) Suppose both $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ are tableau-inconsistent. Let τ_1 be a contradictory tableau for the first set and τ_2 be a contradictory tableau for the second set. Furthermore, we may assume that β_1 occurs at the root of τ_1 and β_2 occurs at the root of τ_2 (the order of introducing propositions is irrelevant). Then the following is a contradictory tableau for $S \cup \{\beta\}$:

$$\begin{array}{cc} \multicolumn{2}{c}{\beta} \\ \beta_1 & \beta_2 \\ \tau_1' & \tau_2' \end{array}$$

where τ_1' is obtained from τ_1 by removing β_1 from the root, and similarly for τ_2'.

The connection between propositional consistency and inconsistency properties is brought out in the following:

Lemma 16.1.6 If $\mathcal{C}$ be a propositional consistency property, then the collection $\mathcal{I}$ of all sets which are *not* $\mathcal{C}$-consistent is a propositional inconsistency property.

Conversely, if $\mathcal{I}$ be a propositional inconsistency property, then the collection $\mathcal{C}$ of all sets which are *not* $\mathcal{I}$-inconsistent is a propositional consistency property.

Proof. Let $\mathcal{C}$ be a propositional consistency property and $\mathcal{I}$ the collection of all sets which are *not* $\mathcal{C}$-consistent. By (C1) no set S containing a propositional symbol and its negation, $\bot$ or $\neg\top$ can be $\mathcal{C}$-consistent, so that any such set S is $\mathcal{I}$-inconsistent. This establishes (I1).

For (I2): If $S \cup \{\alpha_1, \alpha_2\}$ is $\mathcal{I}$-inconsistent, then $S \cup \{\alpha\}$ is not $\mathcal{C}$-consistent, since otherwise by (C2) $S \cup \{\alpha_1, \alpha_2\}$ would be $\mathcal{C}$-consistent.

For (I3): if both $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ are $\mathcal{I}$-inconsistent, then so is $S \cup \{\beta\}$, since otherwise by (C3) at least one of $S \cup \{\beta_1\}$ or $S \cup \{\beta_2\}$ would be $\mathcal{C}$-consistent. □

2 Model Completeness Theorem

This section is dedicated to our main result:

Theorem 16.2.1 (Model Existence Theorem) If $\mathcal{C}$ is a propositional consistency property and $S \in \mathcal{C}$ (that is, S is $\mathcal{C}$-consistent), then S is satisfiable.

Proof. Let $\mathcal{C}$ be a propositional consistency property and $S \in \mathcal{C}$. We will show that S can be extended to a Hintikka set H. It then follows by Hintikka's Lemma 13.4.4 that H is satisfiable, so that S is as well.

The construction of H is analogous to the construction of a finished tableau for S, except that we must use the properties (C1), (C2) and (C3) instead of the rules for constructing tableaux. Let $S = \gamma_1, \gamma_2, \ldots$ be an enumeration of S. We will build a sequence of sets $H_0, H_1, \ldots$ and let $H = \bigcup_n H_n$; is a Hintikka set and $S \subseteq H$. The key is that each H_n will be $\mathcal{C}$-consistent. The construction is in stages n. We will also keep track of a set T of propositions that we have added to S to fulfill the Hintikka conditions. The construction will add only finitely many propositions to S at each stage n.

Stage $n = 0$. Let $H_0 = S$. Note that H_0 is $\mathcal{C}$-consistent, since S is.

Stage $n+1$. Suppose we have constructed H_n which is $\mathcal{C}$-consistent and which contains S. Let $T = \delta_1, \delta_2, \ldots$ be the finite set of propositions we have added to S at earlier stages, so that $H_n = S \cup T$. There are two steps. Step 1: if $\alpha = \gamma_{n+1}$ is type-A then add its components α_1, α_2 to T, so that $S \cup T \cup \{\alpha_1, \alpha_2\}$ is $\mathcal{C}$-consistent by (C2); if $\beta = \gamma_{n+1}$ is type-B, then add β_1 to T if $S \cup T \cup \{\beta_1\}$ is $\mathcal{C}$-consistent, and otherwise add β_2 to T (note that in this latter case $S \cup T \cup \{\beta_2\}$ must be $\mathcal{C}$-consistent by (C3)); if γ_{n+1} is atomic or the negation of an atomic proposition then do nothing. This concludes step 1. Step 2: let δ_i be the least in T such that δ_i has not yet been reduced and is not atomic or the negation of an atomic proposition. Reduce δ_i as was done to γ_{n+1}. This concludes step 2.

After step 2 we have added 0, 1, 2, 3 or 4 propositions to T. Furthermore, $S \cup T$ is $\mathcal{C}$-consistent. Let $H_{n+1} = S \cup T$.

We have constructed a sequence of sets $H_0, H_1, H_2 \ldots$. Let $H = \bigcup_n H_n$. So, $S \subseteq H$. We will show that H is a Hintikka set by showing it satisfies properties (H1)-(H3) from Definition 13.4.3.

(H1). Suppose $P \in H$ and $\neg P \in H$. Then at some large enough stage n, $P, \neg P \in H_n$. But H_n is $\mathcal{C}$-consistent, so this cannot be the case. Similarly, we cannot have $\bot$ or $\neg\top$ in H.

(H2). Suppose $\alpha \in H$ is a type-A proposition with components α_1 and α_2. If $\alpha = \gamma_n \in S$, then α_1 and α_2 were put into H_n at step 1 of stage n, so that $\alpha_1, \alpha_2 \in H$. If $\alpha \in T$, then $\alpha = \delta_i$ for some i (that is, α was the δ_ith proposition added to T). At some sufficiently large stage n, δ_i will be reduced at step 2, so that α_1 and α_2 were placed in H_n, and thus $\alpha_1, \alpha_2 \in H$.

(H3). Suppose β is a type-B proposition with components β_1 and β_2. If $\beta = \gamma_n \in S$, then one of β_1 or β_2 was added to H_n at step 1 of stage n, so that at least one of β_1 or β_2 is in H. If $\beta \in T$, then $\beta = \delta_i$ for some i (that is, β was the δ_ith proposition added to T). At some sufficiently large stage n, δ_i will be reduced at step 2, so that at least one of β_1 or β_2 were placed in H_n, and thus at least one of β_1 or β_2 is in H.

So, $S \subseteq H$ and H is a Hintikka set. H is satisfiable by Hintikka's Lemma 13.4.4, and thus so is S.

□

Remark 16.2.2 Although the construction of the Hintikka set in the Model Completeness Theorem did not appeal to the construction of a semantic tableau, you can view it as the construction of the left-most noncontradictory branch of a semantic tableau for the set S.

The following corollary is the contrapositive form of the Model Existence Theorem and follows from it by Lemma 16.1.6

Corollary 16.2.3 If $\mathcal{I}$ is a propositional inconsistency property and $S \notin \mathcal{I}$, then S is satisfiable.

Remark 16.2.4 One easy application of the Model Existence Theorem is "another" proof of the Strong Completeness Theorem for semantic tableaux.

Corollary 16.2.5 (Strong Completeness Theorem for semantic tableaux) If Γ is tableau-consistent, then Γ is satisfiable.

Tableau-inconsistency is a propositional inconsistency property from Example 16.1.5 above, so that a set that is tableau-consistent is satisfiable by Corollary 16.2.3.

Remark 16.2.6 In Homework 3 you will give a direct proof of the Compactness Theorem which does not depend upon the construction of semantic tableaux, using the following propositional consistency property:

Definition 16.2.7 (Finite Satisfiability) A set Γ of propositions (possibly infinite) is *finitely satisfiable* if every finite subset Γ_0 is satisfiable.

The Compactness Theorem can be stated in the following equivalent form:

Theorem 16.2.8 (Compactness Theorem) Every finitely satisfiable set of propositions is satisfiable.

This is a surprising and powerful result. Let $\Gamma = \gamma_1, \gamma_2, \ldots$ be finitely satisfiable, and for each n and $\Gamma_n = \gamma_1, \ldots, \gamma_n$. So, Γ_1 is satisfiable by some valuation v_1, and Γ_2 is satisfiable by some (possibly different) valuation v_2, and Γ_3 is satisfiable by some (possibly different) valuation v_3, and so on. These valuations may even be inconsistent, in that they disagree on some propositional symbol. But, no matter by the Compactness Theorem, since there does exist a valuation satisfying all of Γ.

We will consider applications in a later lecture.

3 Strong Soundness and Completeness for Natural Deduction

The aim in this section is to prove the following strong version of soundness and completeness for natural deduction:

$$\Gamma \models \beta \quad \text{if and only if } \Gamma \vdash_{\text{nd}} \beta.$$

The proof of both directions is really by contraposition. For this reason we introduce the notion of natural deduction-consistency.

Definition 16.3.1 A set Γ is *natural deduction-consistent* if there is a proposition β for which $\Gamma \nvdash_{\text{nd}} \beta$. Equvalently, there is no proposition α for which $\Gamma \vdash_{\text{nd}} \alpha$ and $\Gamma \vdash_{\text{nd}} \neg\alpha$. (The equivalence is a simple consequence of the $\bot$ introduction and elimination rules.) A set Γ is *natural deduction-inconsistent* if it is not natural deduction-consistent. That is, $\Gamma \vdash_{\text{nd}} \bot$.

We proved the soundness theorem for natural deduction in Lecture, but we repeat the proof here boosted-up to strong soundness. The proof is based on the original system of natural deduction based on the unified notation of Lecture 9, extended in Lecture 14 to include proofs from a set Γ of propositions. An unfinished natural deduction proof from a set Γ is not itself proof from Γ, since there may be undischarged assumptions. The soundness proof must take this into account.

Theorem 16.3.2 (Strong Soundness of Natural Deduction)

$$\Gamma \vdash_{\text{nd}} \beta \quad \text{implies} \quad \Gamma \models \beta$$

Proof. The proof of soundness is by induction on the length of a proof. Consider a proof of β from premises Γ. At a given line of this proof we may have some assumptions $\gamma_1, \ldots, \gamma_n$ still active, not having been discharged, and a proposition Z on the last line. We will associate with this incomplete derivation the assertion $\Gamma, \gamma_1, \ldots, \gamma_k \models Z$. The strategy is to show that this assertion is always correct, for any (possibly) incomplete natural deduction proof. For a completed natural deduction proof whose last line is Z, the assertion $\Gamma \models Z$ is correct, which is what the soundness theorem asserts.

The proof is by induction on the length p of a natural deduction proof. Let the current undischarged assumptions on line p be $\gamma_1, \ldots, \gamma_k$ and let $\Gamma' = \Gamma \cup \{\gamma_1, \ldots, \gamma_k\}$. Let Z be the proposition on line p. The proof proceeds based on the reason Z was introduced into the proof. If $Z \in \Gamma$ has been introduced into the proof, then $\Gamma \models Z$, so that $\Gamma' \models Z$. If Z has been introduced as the assumption of a type-B introduction rule, then $\Gamma', Z \models Z$.

Suppose that line p is the result of a type-A introduction and that $Z = \alpha_i$ $(i = 1, 2)$ is the component of a type-A proposition α, introduced by type-A elimination. Then α occurs on some earlier line n which is still active at line p. By the inductive hypothesis, $\Gamma' \models \alpha$, since the undischarged assumptions when α was introduced on line n must be a subset of $\gamma_1, \ldots, \gamma_k$. Note that the reason for this is that α cannot depend on an assumption which does not occur in the set $\gamma_1, \ldots, \gamma_k$, because line n is still active at line p. Since $\alpha \simeq \alpha_1 \wedge \alpha_2$, we also have $\Gamma' \models \alpha_1$ and $\Gamma' \models \alpha_2$. So the assertion associated with line p is correct.

Suppose $Z = \alpha$ on line p is introduced by type-A introduction from components α_1 and α_2 on lines n_1 and n_2, respectively. Since these earlier lines are still active, each of α_1 and α_2 depend on a set of undischarged assumptions which is a subset of $\gamma_1, \ldots, \gamma_k$. So, by the induction hypothesis, both $\Gamma' \models \alpha_1$ and $\Gamma' \models \alpha_2$. Since $\alpha \simeq \alpha_1 \wedge \alpha_2$ it follows that $\Gamma' \models \alpha$.

Suppose that line p is $Z = \beta_i$ $(i = 1, 2)$, the result of a type-B elimination rule from the type-B proposition β. Then both β and $\neg \beta_{3-i}$ (the other component of β) occur on earlier lines n and m which are still active on line p. By the inductive hypothesis, $\Gamma' \models \beta$ and $\Gamma' \models \neg \beta_{3-i}$, since the undischarged assumptions these depend upon must be a subset of $\gamma_1, \ldots, \gamma_k$. You can verify that the following is a tautology:

$$((P \vee Q) \wedge \neg Q) \rightarrow P),$$

this is the propositional form of the inference $\beta_1 \vee \beta_2, \neg \beta_{3-i} \models \beta_i$, known as *disjunctive syllogism.* The type-B elimination rule is essentially an application of disjunctive syllogism, since $\beta \simeq \beta_1 \vee \beta_2$, so that $\beta, \neg \beta_{3-i} \models \beta_i$. Putting this together with our earlier hypotheses, it follows that $\Gamma' \models \beta_i$.

Suppose that $Z = \beta$ is introduced by type-B introduction with components β_1 and β_2. Suppose that $\neg \beta_i$ $(i = 1, 2)$ was introduced as an assumption on line n and from this assumption β_{3-i} was deduced on line m, immediately to the right of the assumption line introduced by $\neg \beta_i$. The importance of this is that the derivation of β_{3-i} depends only on $\neg \beta_i$, together with the undischarged assumptions when $\neg \beta_i$ was introduced on line n, which are thus $\gamma_1, \ldots, \gamma_k$. The undischarged assumptions on line m must then be $\gamma_1, \ldots, \gamma_k, \neg \beta_i$, so that the assertion $\Gamma', \neg \beta_i \models \beta_{3-i}$ is correct by the induction hypothesis. We want to show that $\Gamma' \models \beta_i \vee \beta_{3-i}$, which together with $\beta \simeq \beta_i \vee \beta_{3-i}$ implies that $\Gamma' \models \beta$. First note that the following is a tautology

$$P \rightarrow (P \vee Q).$$

Since $\Gamma', \neg \beta_i \models \beta_{3-i}$ and $\beta_{3-i} \models (\beta_i \vee \beta_{3-i}$, it follows that $\Gamma', \beta_i \models \beta_i \vee \beta_{3-i}$. We also have by the hypothesis that $\Gamma', \neg \beta_i \models \beta_i \vee \beta_{3-i}$. The final step applies an instance of a valid inference known as *proof by cases*. In proposition form, the instance we need is the following tautology:

$$((P \to Q) \wedge (\neg P \to Q)) \to Q.$$

In our case, we have the following:

$$\begin{array}{lll} (1) & \Gamma' \models \beta_i \to (\beta_i \vee \beta_{3-i}) & \text{and} \\ (2) & \Gamma' \models \neg \beta_i \to (\beta_i \vee \beta_{3-i}) & \text{implies that} \\ (3) & \Gamma' \models \beta_i \vee \beta_{3-i}. & \end{array}$$

We used the Deduction theorem 14.1.3.d to get (1) and (2). Line (3) is exactly what we needed since $\beta \simeq \beta_i \vee \beta_{3-i}$. □

Remark 16.3.3 The more general form of the inference form *proof by cases* can be expressed in propositional form as

$$((P \to R) \wedge (Q \to R)) \to ((P \vee Q) \to R).$$

In words, if you can derive R from P and R from Q, then you can derive R from the weaker hypothesis $P \vee Q$.

The strong soundness theorem establishes that the $\emptyset$ is a natural deduction-consistent set.

Corollary 16.3.4 The natural deduction method is *consistent*. This means that there is no proposition α such that both $\vdash_{\text{nd}} \alpha$ and $\vdash_{\text{nd}} \neg \alpha$.

Proof. Suppose α and $\neg \alpha$ are provable by natural deduction methods. Then by lemma 16.3.2 both α and $\neg \alpha$ are tautologies. But this is impossible since one of α or $\neg \alpha$ must be false under any valuation. So the natural deduction method is consistent. □

For the Strong Completeness Theorem we will use the Model Existence Theorem. A set Γ is natural deduction inconsistent if $\Gamma \vdash_{\text{nd}} \bot$, so it is more natural to prove that this is a propositional inconsistency property.

Lemma 16.3.5 Natural deduction-inconsistency is a propositional inconsistency property. Thus, natural deduction consistency is a propositional consistency property.

Proof. If Γ is a set of propositions which contains a propositional symbol and its negation, $\bot$ or $\neg \top$, then $\Gamma \vdash_{\text{nd}} \bot$. In the first case this is by the $\bot$ introduction rule, and in the last case by the $\top$ introduction rule and $\bot$ introduction.

Let α_1 and α_2 be the components of a type-A proposition α. If $\Gamma \cup \{\alpha_1, \alpha_2\}$ is natural deduction inconsistent, then $\Gamma, \alpha_1, \alpha_2 \vdash_{\text{nd}} \bot$ via some proof $\mathcal{P}$. Then $\Gamma, \alpha \vdash_{\text{nd}} \bot$, by introducing the proof with α and then following this line with $\mathcal{P}$. Any line of $\mathcal{P}$ with α_1 or α_2 we can justify by type-A elimination.

Let β_1 and β_2 be the components of a type-B proposition β. If both $\Gamma \cup \{\beta_1\}$ and $\Gamma \cup \{\beta_1\}$ are natural deduction-inconsistent, then $\Gamma, \beta_1 \vdash_{\text{nd}} \bot$ and $\Gamma, \beta_2 \vdash_{\text{nd}} \bot$ by proof $\mathcal{P}_1$ and $\mathcal{P}_2$. We can convert this to a proof of $\bot$ from $\Gamma \cup \{\beta\}$ as follows:

1	β	
2	$\quad \mid \beta_1$	
$\vdots$	$\quad \mid \mathcal{P}_1$	
n	$\quad \mid \bot$	
$n+1$	$\neg\,\beta_1$	$\neg$I, 2–n
$n+2$	β_2	BE, 1, $n+1$
$\vdots$	$\mathcal{P}_2$	
m	$\bot$	

You can use the methods of Lecture 10 to eliminate the negation introduction rule on line $n+1$ to convert this into a proof using the rules of the unified system. All occurrences of β_1 in $\mathcal{P}_1$ and β_2 in $\mathcal{P}_2$ are justified by reiteration. □

Theorem 16.3.6 (Strong Completeness of Natural Deduction)

$$\Gamma \models \beta \quad \text{implies} \quad \Gamma \vdash_{\text{nd}} \beta$$

Proof. The proof is by contraposition. Suppose $\Gamma \nvdash_{\text{nd}} \beta$. Then $\Gamma, \neg\,\beta \nvdash_{\text{nd}} \bot$: otherwise, we could convert a proof $\mathcal{P}$ of $\Gamma, \neg\,\beta \vdash_{\text{nd}} \bot$ to a proof of $\Gamma \vdash_{\text{nd}} \beta$ as follows:

1	$\quad \mid \neg\,\beta$	
$\vdots$	$\quad \mid \mathcal{P}$	
n	$\quad \mid \bot$	
$n+1$	$\neg\,\neg\,\beta$	$\neg$I, 1–n
$n+2$	β	$\neg$E, $n+1$

Using the methods of Lecture 11 we can eliminate the use of negation introduction on line $n+1$ to a proof in the original unified system.

So, $\Gamma \cup \{\neg\,\beta\}$ is natural deduction-consistent. It follows by Lemma 16.3.5 that natural deduction-consistency is a propositional consistency property, so that $\Gamma \cup \{\neg\,\beta\}$ is satisfiable by the Model Completeness Theorem. Thus, $\Gamma \not\models \beta$. □