1 Strong Soundness and Completeness

Our goal in this section is to strengthen the soundness and completeness theorem from lecture 13:

$$\models \alpha \quad \text{if and only if } \vdash \alpha$$

to tautological consequence: for any set Γ of propositions,

$$\Gamma \models \beta \quad \text{if and only if } \Gamma \vdash \beta.$$

It is useful to rephrase the problem.

Definition 15.1.1 A set Γ of propositions is *tableau-consistent* if there is a proposition β for which $\Gamma \not\vdash \beta$.
The soundness theorem from Lecture 12 established that $\emptyset$ is tableau-consistent.

We will rephrase the strong soundness and completeness theorem in their contrapositive form, which is the form we will prove.

Theorem 15.1.2 (Strong Soundness and Completeness Theorem: Consistency form) A set of propositions Γ is satisfiable if and only if Γ is tableau-consistent.

Corollary 15.1.3 (Strong Soundness and Completeness Theorem: Deduction form) For any set of propositions Γ,

$$\Gamma \models \beta \quad \text{if and only if } \Gamma \vdash \beta.$$

Proof. Note that (1) $\Gamma \models \beta$ if and only if $\Gamma \cup \{\neg \beta\}$ is unsatisfiable, and (2) $\Gamma \vdash \beta$ if and only if $\Gamma \cup \{\neg \beta\}$ is inconsistent. Part (1) follows directly from the definition of tautological consequence: $\Gamma \models \beta$ if and only if there is no valuation v which satisfies Γ and for which $v(\beta) = \mathbf{F}$, that is, $\Gamma \cup \{\neg \beta\}$ is unsatisfiable. Part (2) is even easier: $\Gamma \cup \{\neg \alpha\}$ is inconsistent if and only if there is a contradictory tableau from the premises $\Gamma \cup \{\neg \alpha\}$ with $\top$ at the root. A proof with $\top$ at the root is a proof of a contradiction from $\Gamma \cup \{\neg \alpha\}$ alone. □

We first prove two lemmas which are usefully isolated.

Lemma 15.1.4 If Γ is a (possibly infinite) set of propositions, there is a finished tableau for Γ.
(Note: the tableau need not be finite.)

Proof. The proof is simple, but we need to be a little careful that the tableau we produce in the end is finished. This is also the first time that we will need to construct a tableau which is a limit of a sequence of finite tableaux. The construction is by recursion (on the natural numbers) and produces a sequence of tableaux $\tau_1, \tau_2, \ldots$ according to Definition 14.2.1.

Let $\Gamma = \alpha_1, \alpha_2, \ldots$. Define τ_0 to be the one node tree with α_1 at the root. We will count propositions as we add them to nodes in our tree, so $\alpha_1 = \xi_1$. We need to meet two conditions: every α_i will eventually have to placed on each open branch in the tableaux we construct, as long as the tableaux are non-contradictory. We also need to eventually reduce each proposition ξ_j after they are added to the tableaux.

Suppose we have constructed $\tau_1, \ldots, \tau_n$, and the propositions on the tree are currently numbered $\xi_1, \ldots, \xi_m$. If τ_n is contradictory, then let $\tau_n = \tau_{n+1}$ (we have a finished tree). Otherwise, our task will depend on whether n is even or odd. If $n + 1 = 2k$, then add α_k to each non-contradictory branch, and let this be our new tree τ_{n+1}. If $n + 1 = 2k + 1$ then reduce ξ_j, where j is smallest with ξ_j unreduced on some path, on each path on which it occurs. This will add several new propositions to our list $\xi_{m+1}, \ldots, \xi_p$. Technically, we should reduce ξ_{m+1} on each path through τ_n one-at-a-time, but we will let τ_{n+1} be the result of reducing ξ_{m+1} on each path (which you can think of as a sequence of extension $\tau n_1, \ldots, \tau_{n_i} = \tau_{n+1}$). Note that regardless of whether the stage is even or odd we only add a finite number of propositions to τ.

Let $\tau = \cup_n \tau_n$. Then τ is finished. If π is a non-contradictory path through τ, then α_k was added to π at stage $2k$, so each proposition in Γ is on π. If β is a proposition on π, then $\beta = \xi_k$ for some k (that is, β is the kth node added to τ.) So, β will be reduced as soon as each of $\xi_1, \ldots, \xi_{k-1}$ are reduced. Thus, π is finished. Since π was an arbitrary non-contradictory path on τ, the tableau τ is finished. □

Lemma 15.1.5 Let Γ is a (possibly infinite) set of propositions. If π is a non-contradictory finished path in a tableau (not necessarily finished), then π is a Hintikka set. Thus, Γ is satisfiable.

Proof. π is a Hintikka set follows from Theorem 13.2.1. It follows that there is a valuation v such that $v(\beta) = \mathbf{T}$ for each proposition β on π. Since every proposition in Γ is on π, Γ is satisfiable. □

Strong Soundness and Completeness. The strong soundness theorem is: if Γ is satisfiable, then every tableau constructed from Γ has a non-contradictory path. The argument is essentially the same as the soundness theorem 12.2.3: given a satisfying assignment v for Γ, any tableau must have a path on which v agrees. The only modification here is that a proposition γ from Γ may be added to a path, but v agrees with any such proposition by hypothesis.

The strong completeness theorem is that any set Γ of propositions which has no contradictory tableaux is satisfiable. But by 15.1.4, Γ has a finished tableau with a non-contradictory path π, and by 15.1.5 π is a Hintikka set which contains every proposition of Γ, so that Γ is satisfiable. □

2 König's Lemma

Remark 15.2.1 We clearly near infinite tableau to guarantee that every set of proposition Γ has a finished tableau. We saw an example of this in Lecture 14. For a trivial example any finished tableau for the following set must be infinite:

$$\Gamma = \{P_0, P_1, \ldots, P_n, \ldots\}.$$

In fact, we may restrict *tableaux proofs* to be finite tableaux, even proofs from infinite sets of premises.

The Compactness Theorem is a truly deep theorem, since there is nothing inherent in the method of semantic tableaux which rules out the possibility of infinite proofs from infinite sets of premises. One sign of the "depth" of the theorem is that it has many very surprising consequences, even though you might look at propositional logic as rather weak. We will explore its consequences later. A second sign of its "depth" is that it is highly *nonconstructive*. One sign of this is that theorem offers absolutely no method of constructing a finite proof of $\Gamma \vdash \beta$ when Γ is infinite. It merely says one exists. The theorem can be derived from a general result on trees, known as König's Lemma, named for the first to prove it, Dénés König in 1936 (a decade after the first version of the Compactness Theorem was derived for propositional logic). Later we will see that the Compactness Theorem is equivalent to König's Lemma.

First, some terminology about trees.

Definition 15.2.2 An *infinite* tree τ is a tree with infinitely many nodes. A *finitely branching* tree is one in which every node has finitely many children (although there need not be a bound on the number of children of a node).

All trees we have considered are binary (two) branching trees, so finitely branching.

All our nodes in our trees are assumed to be at some finite level, where the level is the number of ancestors of a node to reach the root. So, a path may be infinite, but every node is at a finite level. This is not essential, but there is no need to consider trees with nodes at transfinite levels.

Theorem 15.2.3 (König's Lemma) If a finitely branching tree τ has infinitely many nodes, then it has an infinite path.

Proof. Let τ be a finitely branching tree with infinitely many nodes. We will define a path $x_1, x_2, x_3, \ldots$ through τ by recursion on the natural numbers. Call a node *rich* if it has infinitely many descendants. Note that the root is rich since τ is infinite. Let x_1 be the root since it is rich and note that the level of x_1 is 1. Suppose we have defined the first n elements of P: $x_1, x_2, \ldots, x_n$, so that node x_i is on the i^{th} level and x_n is rich. Since x_n has only finitely many children, but has infinitely many descendants, some child must have infinitely many descendants, so is rich. Let x_{n+1} be a child of x_n which is rich and on the $n + 1^{\text{st}}$. Let $\pi = \{x_i : i \geq 1\}$. Then π has a node on each finite level, so is a path through τ. $\square$

The next aside is not important in what follows, but only for a little precision.

Aside. We give a more precise definition of the path given by König's lemma. We are defining a function $f : \mathbb{N} \to \mathscr{T}$ so that $f(0) = \lambda$ (the root), and $G : \mathscr{T} \to \mathscr{T}$ is a function defined so that

$$G(x) = \begin{cases} y & \text{for some rich immediate successor } y\text{, if such exists} \\ \text{garbage} & \text{otherwise.} \end{cases}$$

The point is that in the construction of f never accesses "garbage", because $f(n)$ will always have a rich immediate successor to choose. The proof of this is by mathematical induction on n.

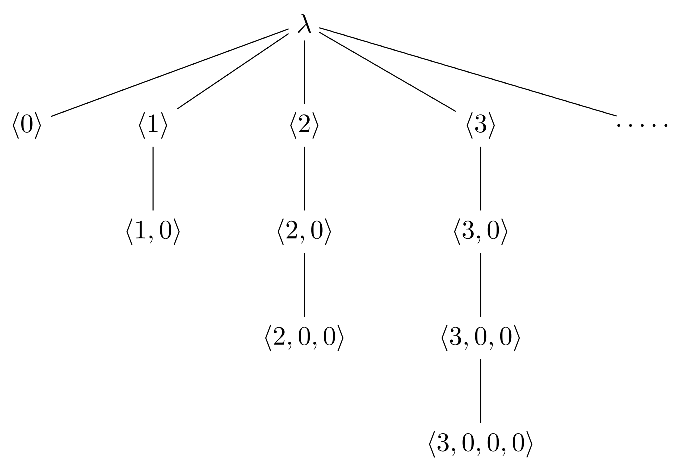
You might notice that I simply said "choose" a rich immediate successor if there is one. In general, there is no principled way of doing this, but we can take our trees to be finite sequences of natural numbers, so that we can order the children from first to last, and so take the first child which is rich.

It is clear that if x is a rich node and has finitely many immediate successors, then it has a rich immediate successor. This is a consequence of the the pigeonhole principle: if there are infinitely many pigeons (successors of x) and finitely many holes (children of x), then alot of pigeons (infinitely many) must be shacking-up in the same hole.

A simple corollary of König's Lemma is the Brouwer Fan Theorem, due to the Dutch intuituitionist logician L.E.B. Brouwer. The Theorem predates König's Lemma.

Theorem 15.2.4 (Brouwer's Fan Theorem) For a finitely branching tree, if all paths are finite, then the tree is finite.

Example 15.2.5 (failure of König's lemma for infinitely branching trees) Here is an example of an infinite tree, with no infinite paths. The only rich node is the root λ. Of course, the tree is infinitely branching, so it runs afoul of König's lemma.



3 Compactness Theorem

The important corollary of König's Lemma is that if a tableau proof exists to establish $\Gamma \vdash \beta$, then a finite tableau proof exists for this.

Corollary 15.3.1 If there is a tableau proof of β from the premises Γ, that is $\Gamma \vdash \beta$, then there is a *finite* tableau proof of β from Γ.

Proof. Let τ be a tableau proof of β from premises Γ. Every path π through τ is contradictory, but it is possible that there are propositions on π following a contradiction. So, let τ' be the tableau obtained from τ by pruning each path π through τ to the point where a contradiction occurs and long enough to ensure each reduced proposition on π up to this point have been fully reduced. (The contradiction could occur while reducing with a type-A rule so that there is a proposition added by the rule after the contradiction has been obtained.)

Each path through τ' is finite because contradictory. Since τ' is binary branching, it follows by Köning's Lemma that τ' is finite. □

As another corollary we get the *Compactness Theorem*:

Corollary 15.3.2 If $\Gamma \models \beta$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ for which $\Gamma_0 \models \beta$.

Proof. Suppose $\Gamma \models \beta$. By the strong completeness theorem, $\Gamma \vdash \beta$, so there is a *finite tableau* τ from Γ with root $\neg \beta$. A finite tableau can only use the Γ-introduction rule a finite number of times, so there is a finite subset Γ_0 of Γ for which τ is a tableau from Γ_0. Thus, $\Gamma_0 \vdash \beta$. By the strong soundness theorem, $\Gamma_0 \models \beta$ □