1 Propositional Consequence: Proof from premises

Remark 14.1.1 Given a set Γ of propositions (the premises or axioms) and proposition β (the consequence or conclusion), we defined in Lecture 4 the notion that Γ tautologically implies β, denoted by $\Gamma \models \alpha$, if every valuation which satisfies Γ makes β true. This usage of implication accords with one feature of implication: if a proposition β is implied by a set of premises Γ, then it should be impossible for all propositions in Γ to be true and yet β false. This is the semantic notion of implication.

We defined in earlier lectures the notion of a tableau proof of a proposition, $\vdash \beta$, which we will extend here to include a proof of a proposition β from a set of premises Γ, denoted by $\Gamma \vdash \beta$. The extension will allow us to introduce propositions from Γ into the construction of a tableau whose root is $\neg \beta$.

We will extend the notion of natural deduction proof to include the use of premises in a later lecture.

We begin with some simple properties of tautological implication. Part (d) was from Homework (2) and part (e) will be in Homework 3.

Lemma 14.1.2

(a) If $\Gamma, \alpha \models \alpha$.

(b) If $\Gamma \models \beta$ and $\Gamma \subseteq \Sigma$, then $\Sigma \models \beta$.

(c) $\Gamma \models \alpha$ and $\Gamma \models \beta$ if and only if $\Gamma \models \alpha \wedge \beta$.

(d) (Deduction Theorem) $\Gamma, \alpha \models \beta$ if and only if $\Gamma \models \alpha \rightarrow \beta$. More generally, $\Gamma, \alpha_1, \ldots, \alpha_n \models \beta$ if and only if $\Gamma \models \alpha_1 \rightarrow (\alpha_2 \rightarrow \ldots (\alpha_n \rightarrow \beta) \ldots)$.

(e) (Cut Theorem) If $\Gamma, \alpha \models \beta$ and $\Gamma \models \alpha$, then $\Gamma \models \beta$. More generally, if $\Gamma, \alpha_1, \ldots, \alpha_n \models \beta$ and $\Gamma \models \alpha_i$ for each $i \leq n$, then $\Gamma \models \beta$

The following corollary reduces the problem of determining whether a proposition β follows from a *finite set* of "axioms" $\alpha_1, \ldots, \alpha_n$ to the problem of determining whether some proposition is a tautology:

Corollary 14.1.3 For any propositions $\alpha_1, \ldots, \alpha_n, \beta$,

$$\alpha_1, \ldots, \alpha_n \models \beta \quad \text{if and only if} \quad \models (\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow \beta.$$

Consequently, when $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$,

$$\Gamma \models \beta \quad \text{if and only if} \quad \vdash (\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow \beta.$$

Proof. First, $\alpha_1, \ldots, \alpha_n \models \alpha_i$ for each $i \leq n$. So, if

$$\alpha_1, \ldots, \alpha_n \models \beta,$$

then n applications of cut 14.3.3.e yields

$$\alpha_1 \wedge \ldots \wedge \alpha_n \models \beta,$$

so by the deduction theorem 14.3.3.d $\models (\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow \beta$.

Conversely, if $\models (\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow \beta$, then by the deduction theorem 14.3.3.c

$$\alpha_1 \wedge \ldots \wedge \alpha_n \models \beta.$$

But,

$$\alpha_1, \ldots, \alpha_n \models \alpha_1 \wedge \ldots \wedge \alpha_n$$

by n applications of 14.3.3.c, so by cut **??**.e

$$\alpha_1, \ldots, \alpha_n \models \beta.$$

The last clause now follows by the soundness and completeness theorems of Lecture 13:

$$\Gamma \models \beta \quad \text{if and only if} \quad \models (\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow \beta \quad \text{if and only if} \quad \vdash (\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow \beta.$$

□

Remark 14.1.4 If we have an infinite set of axioms Γ, then determining whether $\Gamma \models \beta$ cannot be reduced (at least it is not obvious that it can be reduced) to a question of whether some proposition is a tautology or not. It would be useful to extend the methods of proof with semantic tableaux and natural deduction to allow deductions to utilize premises from a set Γ. We first consider an extension of semantic tableaux to include the introduction of premises into the construction.

2 Tableaux Proof from Premises

In a proof of β from a set of propositions Γ we take the propositions of Γ to be among our commitments in the deduction, besides $\neg\beta$ at the root. We first define the notion of a *tableau for* Γ, extending the notion of semantic tableaux from Lecture 8.

Definition 14.2.1 (Tableaux from Premises) Let Γ be a set of propositions. We extend the semantic tableaux rules of Lecture 8 to allow the inclusion of propositions from Γ into a tableau:

> The Γ-*introduction rule for tableaux*: Any member α of Γ can be added to the end of any tableau branch.

A *finite tableau from* Γ is a binary tree, each node labeled with a proposition, and which satisfies the following inductive definition:

(a) All one-node trees labeled with a proposition are finite tableaux.

(b) If τ is a finite tableau, π a path through τ, and A on π, then the extension placing the components A_1 and the A_2 on π is also a finite tableau.

(c) If τ is a finite tableau, π a path through τ, and $\mathcal{B}$ on π, then the extension of π placing the component B_1 on the left branch and the B_2 on the right branch is also a finite tableau.

(d) If τ is a finite tableau, π a path through τ, then the tree obtained by placing a proposition α from Γ onto the end of π is a finite tableau.

If $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ is a (finite or infinite) sequence of finite tableaux from Γ such that for each $n \geq 0$, τ_{n+1} is constructed from τ_n by an application of either (b), (c), or (d), then $\tau = \cup_n \tau$ is a *tableau.*

We will also need to extend the notion of a finished tableau to tableaux constructed from premises. Since a set of premises could be infinite, we will really need to consider infinite tableaux in an attempt to construct a finished tableau from a set Γ.

Definition 14.2.2 (contradictory, finished tableau) Let τ be a tableau from Γ and π a path through τ

- A proposition α on π has been *reduced on* π if one of the following three conditions apply:
  - (i) α is a literal.
  - (ii) $\alpha = A$ and both its A_1 and A_2 components are on π.

(iii) $\alpha = B$ and at least one of its B_1 or B_2 component are on π.

- π is *contradictory* if a proposition and its negation are on π, $\bot$ is on π or $\neg\top$ is on π. π is *finished* if it is contradictory, or every proposition in Γ is on π and every proposition on π has been reduced on π.
- τ is *finished* if every path through τ is finished.
- τ is *contradictory* if every path through τ is contradictory.
- A *tableau proof from premises* Γ of a proposition β is a contradictory tableau from Γ whose root is labeled $\neg\beta$. We denote this by $\Gamma \vdash \beta$.

Example 14.2.3 Here is a tableau proof of $(Q \to R)$ from the set of premises $\{(P \to Q) \to R, P\}$.

(1) $\neg(Q \to R)$
(2) $(P \to Q) \to R$
(3) P
(4) Q (1)
(5) $\neg R$ (1)

Left branch:
(6) $\neg P \to Q$ (2)
(7) P (6)
(8) $\neg Q$ (6)
$\otimes$ (4,8)

Right branch:
(9) R (2)
$\otimes$ (5,9)

A finished semantic tableau from premises can be infinite.

Example 14.2.4 Let Q be a propositional symbol distinct from all those in the set $\{P_1, P_2, P_3, \ldots\}$ of propositional symbols. Lets test whether the following implication holds:

$$\{P_1, (P_1 \to P_2), (P_2 \to P_3), (P_3 \to P_4), \ldots\} \vdash Q.$$

Start the tableau with:

$\neg Q$
P_1
$(P_1 \to P_2)$
Left branch: $\neg P_1$, $\otimes$
Right branch: P_2

Since this tableau is still open and all nodes reduced, add $(B_2 \to B_3)$:

$\neg Q$
P_1
$(P_1 \to P_2)$
Left branch: $\neg P_1$, $\otimes$
Right branch: P_2
$(P_2 \to P_3)$
Left branch: $\neg P_2$, $\otimes$
Right branch: P_3

Every node in this tableau is reduced, but there still has an open branch and unused propositions, so it is not finished. Add $(B_3 \to B_4)$

$$\neg Q$$
$$P_1$$
$$(P_1 \to P_2)$$
$$\neg P_1 \qquad P_2$$
$$\otimes \qquad (P_2 \to P_3)$$
$$\neg P_2 \qquad P_3$$
$$\otimes \qquad (P_3 \to P_4)$$
$$\neg P_3 \qquad P_4$$
$$\otimes$$

We can continue indefinitely, but there will always be an open path. The finished tableau for this set of premises will be infinite, and have the infinite non-contradictory path

$$\pi = \{(P_1 \to P_2), \ldots, (P_i \to P_{i+1}), \ldots, P_1, \ldots, P_i, \ldots, \neg Q\}$$

You can also verify that π is a Hintikka set.

Our notion of tableau-deducibility $\vdash$ shares many of the properties of tautological implication, which are directly verifiable.

Lemma 14.2.5

(a) If $\Gamma, \alpha \vdash \alpha$.

(b) If $\Gamma \vdash \beta$ and $\Gamma \subseteq \Sigma$, then $\Sigma \vdash \beta$.

(c) $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$ if and only if $\Gamma \vdash \alpha \wedge \beta$.

(d) (Deduction Theorem) $\Gamma, \alpha \vdash \beta$ if and only if $\Gamma \vdash \alpha \to \beta$. More generally, $\Gamma, \alpha_1, \ldots, \alpha_n \vdash \beta$ if and only if $\Gamma \vdash \alpha_1 \to (\alpha_2 \to \ldots (\alpha_n \to \beta) \ldots)$.

(e) (Cut Theorem) If $\Gamma, \alpha \vdash \beta$ and $\Gamma \vdash \alpha$, then $\Gamma \vdash \beta$. More generally, if $\Gamma, \alpha_1, \ldots, \alpha_n \vdash \beta$ and $\Gamma \vdash \alpha_i$ for each $i \leq n$, then $\Gamma \vdash \beta$.

Proof. Part (a) follows from the two node tree: $\neg \alpha, \alpha$. For (b), note that a tableau proof from Γ is a tableau proof from Σ, when $\Gamma \subseteq \Sigma$. For (c), let τ be a tableau proof for $\Gamma \vdash \alpha$ and σ be a tableau proof for $\Gamma \vdash \alpha$, then the following is a tableau proof for $\Gamma \vdash \alpha \wedge \beta$:

$$\neg(\alpha \wedge \beta)$$
$$\neg \alpha \qquad \neg \beta$$
$$\tau \qquad \sigma$$

Conversely, in a tableau proof ρ for $\Gamma \vdash \alpha \wedge \beta$ there are three possibilities: (i) the root is never reduced and a contradiction appears because either $(\alpha \wedge \beta)$ appears or $\neg\neg(\alpha \wedge \beta)$ appears, (ii) Γ is contradictory, or (iii) the root is reduced. In cases (i) and (iii) there is a direct tableau proof of α and β, while for case (ii) there is a tableau proof of any proposition.

For (d),l et τ be a tableau proof of $\Gamma\alpha \vdash \beta$. Label the root of a new tableau $\neg(\alpha \to \beta)$ and use the type-A rule so the tableau now looks like:

$$\neg(\alpha \to \beta)$$
$$\alpha$$
$$\neg \beta$$

Now place τ following the third line (with the root $\neg\beta$ removed). Note that it is also possible to remove instances of α from τ, although this is not necessary.

Conversely, any proof of τ of $\Gamma \vdash \alpha \to \beta$ can be turned into a proof of $\Gamma, \alpha \vdash \beta$ by removing $\neg(\alpha \to \beta)$ and beginning the tableau with $\neg\beta, \alpha$ and continuing with τ.

Finally, part (e) is left for Homework 3. □

3 Natural Deduction Proof from Premises

We now extend the notion of a natural deduction proof to a natural deduction proof from premises.

Definition 14.3.1 A natural deduction *proof* of β from a set Γ of propositions is a natural deduction proof, except that on any line of the proof we may introduce a member of Γ. We will write $\Gamma \vdash_{\text{nd}} \beta$ to indicate that there is a proof of β from the premises Γ.

Example 14.3.2 We show $P \vee Q, P \to R, \neg S \to \neg Q \vdash_{\text{nd}} R \vee S$. If you are proving a proposition from a *finite set of premises*, it is a good idea to place the premises at the beginning of the proof (although this is not necessary).

1	$P \vee Q$	
2	$P \to R$	
3	$\neg S \to \neg Q$	
4	│ P	
5	│ R	$\to$E, 2, 4
6	│ $R \vee S$	$\vee$I, 5
7	│ Q	
8	│ │ $\neg S$	
9	│ │ $\neg Q$	$\to$E, 3, 8
10	│ │ $\bot$	$\bot$I, 7, 9
11	│ $\neg\neg S$	$\neg$I, 8–10
12	│ S	$\neg$E, 11
13	│ $R \vee S$	$\vee$I, 12
14	$R \vee S$	$\vee$E, 1, 4–6, 8–13

Our notion of deducibility $\vdash_{\text{nd}}$ shares many of the properties of tautological implication, which are directly verifiable.

Lemma 14.3.3

(a) If $\Gamma, \alpha \vdash_{\text{nd}} \alpha$.

(b) If $\Gamma \vdash_{\text{nd}} \beta$ and $\Gamma \subseteq \Sigma$, then $\Sigma \vdash_{\text{nd}} \beta$.

(c) $\Gamma \vdash_{\text{nd}} \alpha$ and $\Gamma \vdash_{\text{nd}} \beta$ if and only if $\Gamma \vdash_{\text{nd}} \alpha \wedge \beta$.

(d) (Deduction Theorem) $\Gamma, \alpha \vdash_{\text{nd}} \beta$ if and only if $\Gamma \vdash_{\text{nd}} \alpha \to \beta$. More generally, $\Gamma, \alpha_1, \ldots, \alpha_n \vdash_{\text{nd}} \beta$ if and only if $\Gamma \vdash_{\text{nd}} \alpha_1 \to (\alpha_2 \to \ldots(\alpha_n \to \beta)\ldots)$.

(e) (Cut Theorem) If $\Gamma, \alpha \vdash_{\text{nd}} \beta$ and $\Gamma \vdash_{\text{nd}} \alpha$, then $\Gamma \vdash_{\text{nd}} \beta$. More generally, if $\Gamma, \alpha_1, \ldots, \alpha_n \vdash_{\text{nd}} \beta$ and $\Gamma \vdash_{\text{nd}} \alpha_i$ for each $i \leq n$, then $\Gamma \vdash_{\text{nd}} \beta$.

Proof.
(a). This is obvious, simply write down α. For (b), we are simply drawing from a bigger pool of propositions when $\Gamma \subseteq \Sigma$, so a deduction of β from Γ is also one from Σ to Γ. Part (c) follows easily from the introduction and elimination rules for $\wedge$. For part (d), if ρ is a proof of $\Gamma, \alpha \vdash_{\text{nd}} \beta$, then we can begin a proof of $\Gamma \vdash_{\text{nd}} \alpha \rightarrow \beta$ by

$$
\begin{array}{l|l}
1 & \begin{array}{|l} \alpha \\ \hline \rho \\ \beta \end{array} \\
\vdots & \\
n & \\
n+1 & \alpha \rightarrow \beta
\end{array}
$$

Instance of α in ρ are justified by the reiteration rule from line 1.
Conversely, from a proof of ρ $\Gamma \vdash \alpha \rightarrow \beta$ begin with the proof ρ of $\alpha \rightarrow \beta$:

$$
\begin{array}{l|ll}
\vdots & \rho & \\
n & \alpha \rightarrow \beta & \\
n+1 & \alpha & \\
n+1 & \beta & \rightarrow\text{E}, n, n+1
\end{array}
$$

Finally, for (e) if $\Gamma, \alpha \vdash \beta$ and $\Gamma \vdash \alpha$, then start a proof of α from Γ, then place after this a proof of β from Γ, α, but now you can justify each instance of α in this proof using the reiteration rule. □

4 Soundness and Completeness

The *strong soundness and completeness* theorems connect tautological implication and proof from premises:

$$
\begin{array}{lll}
\Gamma \models \beta & \text{if and only if} & \Gamma \vdash \beta \\
\Gamma \models \beta & \text{if and only if} & \Gamma \vdash_{\text{nd}} \beta.
\end{array}
$$

Both equivalences are true, however there is a significant difference between the statements. The notion of proof for natural deduction systems are *necessarily finite*; but tableaux can be infinite, so it is open whether tableaux proofs from a set of premises must always be finite as well. This is a problem which is as hard as showing the strong completeness of the natural deduction system. To see this, suppose $\Gamma \models \beta$. Then by strong completeness for natural deduction, $\Gamma \vdash_{\text{nd}} \beta$. Since a finite natural deduction proof can only use finitely many premises, so the following is true:

$$\Gamma \vdash_{\text{nd}} \beta \quad \text{implies for some finite } \Gamma_0, \quad \Gamma_0 \vdash_{\text{nd}} \beta.$$

By strong soundness it follows that $\Gamma_0 \models \beta$. It is worth stating this connection separately for tautological consequence

$$\Gamma \models \beta \quad \text{implies for some finite } \Gamma_0, \quad \Gamma_0 \models \beta.$$

This is known as the *Compactness Theorem* and is a deep result with many surprising consequences. We will explore this later.

For now we note only the following easy fact about tableaux proofs:

Lemma 14.4.1 If Γ is a finite set of propositions and $\Gamma \vdash \beta$, then there is a finite tableau proof of this.

Proof. Let $\Gamma = \alpha_1, \ldots \alpha_n$ If $\Gamma \models \beta$ then there is a finite contradictory tableau for $\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \neg\beta$ by Lemma 8.3.5 and the Deduction theorem for tableaux **??**. □