1 Soundness Theorem for Semantic Tableaux

Theorem 12.1.1 (soundness of the tableau method) If α is tableau provable, then α is a tautology, i.e.

$$\vdash \alpha \quad \text{implies} \quad \models \alpha$$

Proof. Let τ be any finished (possibly infinite) tableau with root $\neg\alpha$. Let $\tau = \cup_n \tau_n$ where each τ_n is a finite tableaux and τ_{n+1} extends τ_n by application of a rule to one proposition extending one branch. A valuation v *agrees* with a proposition γ if $v(\gamma) = \mathbf{T}$. A valuation v *agrees with a path* π if v agrees with every proposition on π.

We will prove the contrapositive: If α is not a tautology, then there is some finished non-contradictory path in τ. Since α is not a tautology, there is a valuation v such that $v(\alpha) = \mathbf{F}$, that is, $v(\neg\alpha) = \mathbf{T}$, so that . v agrees with the root $\neg\alpha$. We will construct a path π through τ such that v agrees with every proposition β on π. It then follows that τ cannot be contradictory (namely, π is a finished non-contradictory path). Since τ was an arbitrary tableau for α, α cannot be tableau provable.

Since $\tau = \bigcup_n \tau_n$, we will construct the path π through τ by recursion on n. We will make sure that π_n is a path through τ_n on which v agrees. Then $\pi = \bigcup_n \pi_n$ will be a path through τ on which v agrees.

τ_0 is the one-node tableau labeled with $\neg\alpha$ and $\pi_0 = \neg\alpha$ is a path through τ_0 on which v agrees.

Suppose that π_n through τ_n on which v agrees. If π_n is not extended in τ_{n+1} then let $\pi_n = \pi_{n+1}$. Otherwise τ_{n+1} extends τ_n on π_n, so that there is a proposition γ on π_n which determines the rule extending π_n. There are two possibilites:

γ is type-A and has components γ_1 and γ_2 added to π_n. SInce $\gamma \simeq \gamma_1 \wedge \gamma_2$, and v agrees with γ, $v(\gamma_1 \wedge \gamma_2) = \mathbf{T}$, so that v agrees with γ_1 and γ_2. Let $\pi_{n+1} = \pi_n \cup \{\gamma_1, \gamma_2\}$, so that v agrees with π_{n+1} and this is a path through τ_{n+1}.

γ is type-B and π_n branches in τ_{n+1} into a left branch with component γ_1 and a right branch with component γ_2. Since $\gamma \simeq \gamma_1 \vee \gamma_2$ and v agrees with γ, $v(\gamma_1 \vee \gamma_2)$, and so v agrees with at least one of γ_1 or γ_2. Let $\pi_{n+1} = \pi_n \cup \{\gamma_i\}$ where $i = 0$ if v agrees with γ_1 and $i = 2$ otherwise, that is v agrees with γ_2. So, v agrees with π_{n+1} which is a path through τ_{n+1}. ☐

Corollary 12.1.2 The tableau method is *consistent*. This means that there is no proposition α such that both $\vdash \alpha$ and $\vdash \neg\alpha$.

Proof. Suppose α and $\neg\alpha$ have closed tableaux. Then by lemma 12.2.3 both α and $\neg\alpha$ are tautologies. But any valuation must make either α true or $\neg\alpha$ true. So the tableau method is consistent. ☐

2 Soundness Theorem for Natural Deduction

We now turn to the soundness theorem for the natural deduction system based on uniform notation in Lecture 9. The extended system with the derived rules will then be sound since no new theorems are provable using these derived rules.

Unfinished tableaux are still tableaux themselves. However, an unfinished natural deduction proof is not a proof itself, since it has undischarged assumptions. The soundness proof must take this into account.

First, some simple facts about tautological consequence.

Lemma 12.2.1

(a) If $\Gamma \models \alpha$ and $\Gamma \subseteq \Sigma$, then $\Sigma \models \alpha$.

(b) $\Gamma \models \alpha$ and $\Gamma \models \beta$ if and only if $\Gamma \models \alpha \wedge \beta$.

(c) (Deduction Theorem) $\Gamma, \alpha \models \beta$ if and only if $\Gamma \models \alpha \rightarrow \beta$. More generally, $\Gamma, \alpha_1, \ldots, \alpha_n \models \beta$ if and only if $\Gamma \models \alpha_1 \rightarrow (\alpha_2 \rightarrow \ldots (\alpha_n \rightarrow \beta) \ldots)$.

(d) $\Gamma, \alpha \models \beta$ and $\Gamma, \beta \models \alpha$ if and only if $\Gamma \models \alpha \leftrightarrow \beta$.

(e) (Cut Theorem) If $\Gamma, \alpha \models \beta$ and $\Gamma \models \alpha$, then $\Gamma \models \beta$. More generally, if $\Gamma, \alpha_1, \ldots, \alpha_n \models \beta$ and $\Gamma \models \alpha_i$ for each $i \leq n$, then $\Gamma \models \beta$

Remark 12.2.2 All parts of the lemma follow easily from the definition of tautological consequence given in Definition 4.1.12. Part (c) was on homework 2 and (e) will be on homework 3. Part (d) follows immediately from parts (b) and (c).

It is worth noting that parts (b), (c) and (d) show that there is a close relationship between tautological implication and the connectives $\wedge, \rightarrow, \leftrightarrow$. This does not extend to disjunction (which was on homework 2) nor to negation. The following is not true

- $\Gamma \not\models \alpha$ if and only if $\Gamma \models \neg\alpha$.

You can verify that the implication ($\Rightarrow$) is true, provided Γ is consistent. Of course, if Γ is inconsistent, then $\Gamma \models \alpha$ for every proposition α. The implication ($\Leftarrow$) also fails: let $\Gamma = \{Q\}$, so $Q \not\models P$, however it is also true that $Q \not\models \neg P$.

On to soundness:

Theorem 12.2.3 (soundness of the tableau method) If α is provable in the natural deduction system, then α is a tautology, i.e.

$$\vdash_{\text{nd}} \alpha \quad \text{implies} \quad \models \alpha$$

Proof. The proof of soundness is by induction on the length of a proof. But first we must address the problem of premises which have not yet been discharged. Suppose that at some point in a proof we have assumptions $\gamma_1, \ldots, \gamma_n$ still active, not having been discharged, and a proposition Z on the last line. We will associate with this in complete derivation the assertion $\gamma_1, \ldots, \gamma_k \models Z$, that is, $\{\gamma_1, \ldots \gamma_k\}$ tautologically implies Z. We will show that this assertion is always correct, for any (possibly) incomplete natural deduction proof. For a completed natural deduction proof whose last line is Z, the assertion $\models Z$ is correct, which is what the soundness theorem asserts.

The proof is by induction on the length p of a natural deduction proof. Suppose that Z is introduced by the introduction of a hypothesis (for a type-B derivation rule). If $\gamma_1, \ldots, \gamma_k$ are the undischarged assumptions before line n, then since Z is the proposition on line p introduced as an assumption, the assertion $\gamma_1, \ldots, \gamma_k, Z \models Z$ is correct.

Suppose that line p is the result of a type-A introduction or elimination rule and $\gamma_1, \ldots, \gamma_k$ are the undischarged assumption at line p. Suppose $Z = \alpha_i$ $(i = 1, 2)$ is the component of a type-A proposition α, introduced by type-A elimination. Then α occurs on some earlier line n which is still active at line p. By the inductive hypothesis, $\gamma_1, \ldots, \gamma_k \models \alpha$, since α undischarged assumptions on line n must be a subset of $\gamma_1, \ldots, \gamma_k$. Note that the reason for this is that α cannot depend on an assumption which does not occur in this set, since line n is still active at line p. Since $\alpha \simeq \alpha_1 \wedge \alpha_2$, we also have $\gamma_1, \ldots, \gamma_k \models \alpha_1$ and $\gamma_1, \ldots, \gamma_k \models \alpha_2$. So the assertion associated with line p is correct. Suppose $Z = \alpha$ is introduced by type-A introduction from components α_1 and α_2 on lines n_1 and n_2, respectively. Since these earlier lines are still active, each of α_1 and α_2 depend on a set of undischarged assumptions which is a subset of $\gamma_1, \ldots, \gamma_k$. So, by the induction hypothesis, both $\gamma_1, \ldots, \gamma_k \models \alpha_1$ and $\gamma_1, \ldots, \gamma_k \models \alpha_2$. Since $\alpha \simeq \alpha_1 \wedge \alpha_2$ it follows that $\gamma_1, \ldots, \gamma_k \models \alpha$.

Suppose that line p is the result of a type-B introduction or elimination rule and $\gamma_1, \ldots, \gamma_k$ are the undischarged assumption at line p. Suppose $Z = \beta_i$ $(i = 1, 2)$ is the component of a type-B proposition β, introduced by type-B elimination. Then both β and $\neg\beta_{3-i}$ (the other component of β) occur on earlier lines n and m which are still active on line p. By the inductive hypothesis, $\gamma_1, \ldots, \gamma_k \models \beta$ and $\gamma_1, \ldots, \gamma_k \models \neg\beta_{3-i}$, since the undischarged assumptions these depend upon must be a subset of $\gamma_1, \ldots, \gamma_k$. You can verify that the following is a tautology:

$$((P \vee Q) \wedge \neg Q) \rightarrow P),$$

this is the propositional form of the inference $\beta_1 \vee \beta_2, \neg\beta_{3-i} \models \beta_i$, known as *disjunctive syllogism*. The type-B elimination rule is essentially an application of disjunctive syllogism, since $\beta \simeq \beta_1 \vee \beta_2$, so that $\beta, \neg\beta_{3-i} \models \beta_i$. Putting this together with our earlier hypotheses, it follows that $\gamma_1, \ldots, \gamma_k \models \beta_i$, so the assertion at line p is correct.

Suppose that $Z = \beta$ is introduced by type-B introduction with components β_1 and β_2. Suppose that $\neg\beta_i$ $(i = 1, 2)$ was introduced as an assumption on line n and from this assumption β_{3-i} was deduced on line m, immediately to the right of the assumption line introduced by $\neg\beta_i$. The importance of this is that the derivation of β_{3-i} depends only on $\neg\beta_i$, together with the undischarged assumptions when $\neg\beta_i$ was introduced on line n. Let $\gamma_1, \ldots, \gamma_k$ be the undischarged assumptions on line p. The undischarged assumptions on line m must then be $\gamma_1, \ldots, \gamma_k, \neg\beta_i$, so that the assertion $\gamma_1, \ldots, \gamma_k, \neg\beta_i \models \beta_{3-i}$ is correct by the induction hypothesis. We want to show that $\gamma_1, \ldots, \gamma_k \models \beta_i \vee \beta_{3-i}$, which together with $\beta \simeq \beta_i \vee \beta_{3-i}$ implies that $\gamma_1, \ldots, \gamma_k \models \beta$. First note that the following is a tautology

$$P \rightarrow (P \vee Q).$$

Since $\gamma_1, \ldots, \gamma_k, \neg\beta_i \models \beta_{3-i}$ and $\beta_{3-i} \models (\beta_i \vee \beta_{3-i}$, it follows that $\gamma_1, \ldots, \gamma_k, \beta_i \models \beta_i \vee \beta_{3-i}$. We also have by the hypothesis that $\gamma_1, \ldots, \gamma_k, \neg\beta_i \models \beta_i \vee \beta_{3-i}$. The final step applies an instance of a valid inference known as *proof by cases*. In proposition form, the instance we need is the following tautology:

$$((P \rightarrow Q) \wedge (\neg P \rightarrow Q)) \rightarrow Q.$$

In our case, we have the following:

(1) $\quad \gamma_1, \ldots, \gamma_k \models \beta_i \rightarrow (\beta_i \vee \beta_{3-i})$ and
(2) $\quad \gamma_1, \ldots, \gamma_k \models \neg\beta_i \rightarrow (\beta_i \vee \beta_{3-i})$ implies that
(3) $\quad \gamma_1, \ldots, \gamma_k \models \beta_i \vee \beta_{3-i}$.

We used the Deduction theorem (12.2.1.(c)) to get (1) and (2). Line (3) is exactly what we needed since $\beta \simeq \beta_i \vee \beta_{3-i}$. □

Remark 12.2.4 The more general form of the inference form *proof by cases* can be expressed in propositional form as

$$((P \rightarrow R) \wedge (Q \rightarrow R)) \rightarrow ((P \vee Q) \rightarrow R).$$

In words, if you can derive R from P and R from Q, then you can derive R from the weaker hypothesis $P \vee Q$.

The proof of the following is exactly the same as for 12.1.2.

Corollary 12.2.5 The natural deduction method is *consistent*. This means that there is no proposition α such that both $\vdash_{\text{nd}} \alpha$ and $\vdash_{\text{nd}} \neg\alpha$.