1 Putting the "Natural" in Natural Deduction: Derived Rules

Remark 10.1.1 The natural deduction rules introduced in Definition 9.2.3 are slick, and will be more conducive to proving the soundness and completeness of the natural deduction system, but they reduce all reasoning to conjunctive or disjunctive reasoning, so are far from natural.

A rule is *derived* if the addition does not change the strength of the system. For example, adding the splitting rule to the analytic tableaux proof system (introduced in Definition 8.6.2 of Lecture 8) is a derived rule for the system. However, it is far from obvious this is so, since there is not principled way of converting a proof using the splitting rule to one which does not use it.

The derived rules for the natural deduction system provide convenient "macros" for doing natural deduction proofs, in the sense that you will be able to convert a proof using derived rules into a proof using the original rules from Definition 9.2,3. Each connective will have an introduction (for deriving a proposition of this type) and an elimination rule (for using a proposition of this type). Furthermore, these rules will correspond to more natural patterns of inference.

Definition 10.1.2 (Rules for Natural Deduction)

The rules only apply to propositions which are still active on the line it is applied. The rule of Reiteration and the rules for the constants remain the same.

Negation Rules. The negation elimination rule is simply the A-elimination rule as before. There is a corresponding negation introduction rule. The idea here is that we may deduce that $\neg\,\alpha$ *must hold*, if it is impossible for α to be true. That is, on the assumption of α we can derive a contradiction.

$$\begin{array}{lll} n & \neg\,\neg\,\alpha & \\ \vdots & \vdots & \\ m & \alpha & \neg\text{E}, n \end{array} \qquad \begin{array}{l|ll} n & \alpha & \\ \hline \vdots & \vdots & \\ m & \bot & \\ \end{array} \\ \begin{array}{lll} m+1 & \neg\,\alpha & \neg\text{I}, n\text{–}m \end{array}$$

Conjunction Rules. The type-A rules are natural for conjunctions, so we take them over:

$$\begin{array}{lll} n & \alpha \wedge \beta & \\ \vdots & \vdots & \\ m & \alpha & \wedge\text{E}, n \end{array} \qquad \begin{array}{lll} n & \alpha \wedge \beta & \\ \vdots & \vdots & \\ m & \beta & \wedge\text{E}, n \end{array} \qquad \begin{array}{lll} n & \alpha & \\ \vdots & \vdots & \\ m & \beta & \\ \vdots & \vdots & \\ p & \alpha \wedge \beta & \text{AI}, n, m \end{array}$$

Conditional Rules. These were stated earlier:

$$
\begin{array}{lll}
n & \alpha & \\
\vdots & \vdots & \\
m & \alpha \rightarrow \beta & \\
\vdots & \vdots & \\
p & \beta & \rightarrow\text{E}, m, n
\end{array}
\qquad
\begin{array}{l|l}
n & \alpha \\ \cline{2-2}
\vdots & \vdots \\
m & \beta
\end{array}
\qquad
\begin{array}{lll}
m+1 & \alpha \rightarrow \beta & \rightarrow\text{I}, n\text{–}m
\end{array}
$$

There is a second form of →E suggested by the old B-elimination rule for the conditional:

$$
\begin{array}{lll}
n & \neg\beta & \\
\vdots & \vdots & \\
m & \alpha \rightarrow \beta & \\
\vdots & \vdots & \\
p & \neg\alpha & \rightarrow\text{E}, m, n
\end{array}
$$

This rule corresponds to a type of inference known as *modus tollens.* This rule is not so commonly used directly, instead contraposition is used to convert $\alpha \rightarrow \beta$ to $\neg\beta \rightarrow \neg\alpha$ and then *modus ponens*, which is simply →E above.

Disjunction Rules. The disjunction rules could be take over the type-B introduction and elimination rules, since these more naturally correspond to reasoning using disjunctions. However, the rules are still a little convoluted because of all the negations. So, more common to natural deduction systems are the following rules for disjunction:

$$
\begin{array}{l|lll}
n & \alpha \vee \beta & & \\
m_1 & \quad | & \alpha & \\
\vdots & \quad | & \vdots & \\
m_2 & \quad | & \gamma & \\
p_1 & \quad | & \beta & \\
\vdots & \quad | & \vdots & \\
p_2 & \quad | & \gamma & \\
p_2+1 & \gamma & & \vee\text{E}, n, m_1\text{–}m_2, p_1\text{–}p_2
\end{array}
\qquad
\begin{array}{lll}
n & \alpha & \\
\vdots & \vdots & \\
m & \alpha \vee \beta & \vee\text{I}, n
\end{array}
\qquad
\begin{array}{lll}
n & \beta & \\
\vdots & \vdots & \\
m & \alpha \vee \beta & \vee\text{I}, n
\end{array}
$$

The ∨E rule is a common form of reasoning known as *proof by cases.* If you know $\alpha \vee \beta$ *must be true* and you can derive γ from α and γ from β, then γ *must also be true.* The ∨I rules are pretty clear, but they are rarely used in natural reasoning because $\alpha \vee \beta$ is a weaker conclusion than α. It is more common to derive disjunctions as in the BI rule: if we can derive β from the assumption α is false, then $\alpha \vee \beta$ must be true. In this case, it may not be true that α must hold or β must hold, individually.

2 Deriving the derived rules

Remark 10.2.1 (Theorem introduction) Here is the simplest example of a derived rule: if $\vdash_{\text{nd}} \alpha$, then you may simply write down α on any line of the proof. This rule does not extend what we can derive, because we can always replace α by the proof of α using the natural deduction rules. You can think of α as a macro for a proof of α. This is a commonly used principle in reasoning, and has no equivalent in proof by semantic tableaux.

Example 10.2.2 (Conditional Rules) The rule of $\rightarrow$E is nearly in the form of the BE rule. We can derive the $\rightarrow$E as follows:

n	α	
$\vdots$	$\vdots$	
m	$\alpha \rightarrow \beta$	
$\vdots$	$\vdots$	
p	$\neg\neg\alpha$	AI, n
$p+1$	β	BE, m, p

Note that the components of the type-B proposition $\alpha \rightarrow \beta$ are $\beta_1 = \neg\alpha$ and $\beta_2 = \beta$, so we have essentially replaced $\rightarrow$E by a type-B elimination. Most of the derived rules involve eliminating an application of the $\neg$I rule, as in line (5) here.

The $\rightarrow$I rule above is simply the type-B introduction rule of old, except that our hypothesis is α and not $\neg\neg\alpha$, which it would be by the old rules. That is, by the old rule for type-B introduction for $\rightarrow$:

n	$\neg\neg\alpha$	
$n+1$	α	AE, n
$\vdots$	$\vdots$	
m	β	
$m+1$	$\alpha \rightarrow \beta$	BI, n–m

We have access to α from our assumption $\neg\neg\alpha$ by a type-A elimination, so we might as well allow ourselves to assume α, since this is the hypothesis we typically will use in the deduction. Most of our examples in Lecture 9, that introduced the hypothesis $\neg\neg\alpha$ eliminated the double negation on the very next step.

Example 10.2.3 ($\top$ and $\neg\bot$ equivalence) I'll derive the following tautology $\top \leftrightarrow \neg\bot$. It is an illustration of the use of the rules for these constants.

1	$\neg\neg\bot$	
2	$\bot$	AE, 1
3	$\neg\top$	$\bot$E, 2
4	$\top \rightarrow \neg\bot$	BI, 1–3

1	$\neg\neg\neg\bot$	
2	$\top$	$\top$I, 2
3	$\neg\bot \rightarrow \top$	BI, 1–2

I do not think the direction $\top \rightarrow \neg\bot$ can be proven using just the rules for constants and the $\rightarrow$ rules alone. It can be proven using the $\neg$I rule, but I will use the equivalence to justify that rule. The proof of $\top \rightarrow \neg\bot$ means that I may always simple add $\neg\bot$ to any line of a proof, just as in the $\top$I rule. I could replace this with the above proof of $\top \rightarrow \neg\bot$ and $\top$, followed by $\neg\bot$, justified by $\leftrightarrow$E.

Example 10.2.4 (Negation Rules) Only the negation introduction rule needs to be derived. This is a bit more complicated, but the idea is that $(\alpha \rightarrow \bot) \leftrightarrow \neg\alpha$ can be derived using the old rules. Here is the proof:

1	$\neg\neg(\alpha \to \bot)$	
2	$\alpha \to \bot$	AE, 1
3	$\neg\bot$	$\top$I, 3
4	$\neg\alpha$	BE, 2, 3
5	$(\alpha \to \bot) \to \neg\alpha$	BI, 1–4

1	$\neg\neg\alpha$	
2	α	AE, 1
3	$\quad \neg\alpha$	
4	$\quad \alpha$	R, 2
5	$\quad \bot$	$\bot$I, 3, 4
6	$\alpha \to \bot$	BI, 3–5
7	$\neg\alpha \to (\alpha \to \bot$	BI, 1–6

Line (3) is justified because $\top \leftrightarrow \neg\bot$ is derivable, as in Example **??**.

Now, the $\neg$I can be derived:

n	α	
$\vdots$	$\vdots$	
m	$\bot$	
$m+1$	$\alpha \to \bot$	$\to$I, n–m
$m+2$	$(\alpha \to \bot) \to \neg\alpha$	
$m+3$	$\neg\alpha$	$\to$E, $m+2$, $m+1$

Line $m+2$ is our previous theorem (see Remark **??** for introducing previously proven propositions into a proof).

Example 10.2.5 (Disjunction rules) The rule of $\vee$I is very easy to justify by type-B introduction:

n	α	
$\vdots$	$\vdots$	
m	$\quad \neg\alpha$	
$m+1$	$\quad \alpha$	R, n
$m+2$	$\quad \bot$	$\bot$I, m, $m+1$
$m+3$	$\quad \beta$	$\bot$E, $m+2$
$m+4$	$\alpha \vee \beta$	BI, m–$m+3$

We can introduce $\alpha \vee \beta$ on any line in which α is active. A similar argument shows we can introduce $\alpha \vee \beta$ on any line in which β is active.

The $\vee$E rule can be justified by proving the following tautology:

$$((\alpha \to \gamma) \wedge (\beta \to \gamma)) \to ((\alpha \vee \beta) \to \gamma).$$

I am going to prove this theorem using only the original natural deduction rules, and without using the derived rules.

$$
\begin{array}{lll}
1 & \neg\neg((\alpha \to \gamma) \wedge (\beta \to \gamma)) & \\
2 & (\alpha \to \gamma) \wedge (\beta \to \gamma) & \\
3 & \alpha \to \gamma & A\text{E}, 1 \\
4 & \beta \to \gamma & A\text{E}, 1 \\
5 & \quad \neg\gamma & \\
6 & \quad \neg\alpha & B\text{E}, 3, 5 \\
7 & \quad \neg\beta & B\text{E}, 4, 5 \\
8 & \quad \neg(\alpha \vee \beta) & A\text{I}, 6, 7 \\
9 & \alpha \vee \beta) \to \gamma & B\text{I}, 5\text{–}8 \\
10 & ((\alpha \to \gamma) \wedge (\beta \to \gamma)) \to ((\alpha \vee \beta) \to \gamma) & B\text{I}, 1\text{–}9
\end{array}
$$

Now, $\vee$I is easy to justify:

$$
\begin{array}{lll}
n & \alpha \vee \beta & \\
m_1 & \quad \alpha & \\
\vdots & \quad \vdots & \\
m_2 & \quad \gamma & \\
m_2 + 1 & \alpha \to \gamma & \to\text{I}, m_1\text{–}m_2 \\
p_1 & \quad \beta & \\
\vdots & \quad \vdots & \\
p_2 & \quad \gamma & \\
p_2 + 1 & \beta \to \gamma & \to\text{I}, m_1\text{–}m_2 \\
p_2 + 2 & (\alpha \to \gamma) \wedge (\beta \to \gamma) & \wedge\text{I}, m_2 + 1, p_2 + 1 \\
p_2 + 3 & ((\alpha \to \gamma) \wedge (\beta \to \gamma)) \to ((\alpha \vee \beta) \to \gamma) & \\
p_2 + 4 & (\alpha \vee \beta) \to \gamma & \to\text{E}, p_2 + 3, p_2 + 2 \\
p_2 + 5 & \gamma & \to\text{E}, p_2 + 4, n
\end{array}
$$

Remark 10.2.6 (Reductio ad absurdam) One of the difficulties in using natural deduction is that if you are absolutely stuck in a proof, you can always try to prove something by *reductio ad absurdam*: assume its negation and try to derive a contradiction. This can be formulated as a derived rule:

$$
\begin{array}{ll}
n & \quad \neg\alpha \\
\vdots & \quad \vdots \\
m & \quad \bot \\
m + 1 & \alpha
\end{array}
$$

This overly used principle has the following unhappy disadvantage: it leaves you completely clueless how to derive a contradiction. Proof by contradiction is one of the most popular forms of argument, and is *almost* *never* required, and rarely shortens a proof.

Here is an application which is required, in the proof of the contraposition law:

$$(\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)$$

1	$\neg Q \rightarrow \neg P$	
2	$\quad P$	
3	$\quad\quad \neg Q$	
4	$\quad\quad \neg P$	$\rightarrow$E, 1, 4
5	$\quad\quad P$	R, 2
6	$\quad\quad \bot$	$\bot$I, 4, 5
7	$\quad \neg\neg Q$	$\neg$I, 3–6
8	$\quad Q$	$\neg$E, 7