Math481 Midterm 2 November 23

Name.

Question	Points	Score
1	10	
2	10	
3	20	
4	10	
5	15	
6	10	
Total	75	

1. Translation

Use the following translation key:

- Domain: All things (including moments in time)
- $P(x)$: x is a person.
- $T(x)$: x is a moment in time.
- $F(x, y)$: x can be fooled at time y.

Translate the following

(a) You can fool some of the people all of the time.

(b) You can fool all of the people some of the time.

(c) You can't fool all the people all of the time.

Answer. There is some ambiguity in the translation due to the difficulty in English in expressing nested quantifiers.

(a). Two possible answers:

$$(\forall x)(\exists y)\big(T(x) \wedge P(y) \wedge F(x, y)\big)$$
$$(\exists y)\big(P(y) \wedge (\forall x)(T(x) \rightarrow F(x, y))\big)$$

I think the first is closer to the intended interpretation. It allows that each time x there is someone who can be fooled. The second interpretation insists that it is the same person who is fooled at each time x.

(b). Two possible answers:

$$(\forall y)\big(P(y) \rightarrow (\exists x)(T(x) \wedge F(x, y))\big)$$
$$(\exists x)\big(T(x) \wedge (\forall y)(P(y) \rightarrow F(x, y))\big)$$

I think the first is closer to the intended interpretation. It allows that each person y is fooled at some time. The second interpretation insists that it is the same time at which each person is fooled.

(c). One possibility here:

$$\neg (\forall x)(\forall y)\big((T(x) \wedge P(y)) \rightarrow F(x, y)\big)$$

2. Interpretation in a Structure

The following is a structure $\mathcal{A}$:

- Domain: a,b,c
- $M^{\mathcal{A}} = \{b\}$
- $N^{\mathcal{A}} = \{b, c\}$
- $P^{\mathcal{A}} = \{a, b, c\}$.
- $Q^{\mathcal{A}} = \emptyset$
- $R^{\mathcal{A}} = \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$
- $h^{\mathcal{A}} = a$

Determine whether each of the following are true or false in $\mathcal{A}$.

(a) $(\forall x)\big(R(h, x) \rightarrow N(x)\big)$

(b) $(\forall x)(\exists y)R(x, y)$

(c) $(\forall x)(\forall y)\big(R(y, x) \lor R(x, y)\big)$

(d) $(\forall x)(Q(x) \rightarrow (\exists y)R(y, y))$

(e) $(\forall x)(\forall y)(\forall z)\big((R(x, y) \land R(y, z)) \rightarrow R(x, z)\big)$

Answer.

(a). True.

(b). False. For $x = c$, it is not true that $(\exists y)R(c, y)$.

(c). False. For example, for $x = y = a$, it is not true that $R(a, a)$.

(d). True. Q is an empty predicate, so the conditional $Q(x) \rightarrow (\exists y)R(y, y)$ is true for any choice of x.

(e) True.

3. First-Order Validity

Let β be a sentence in which the constant c does not occur and let x be a variable freely substitutable for c in the sentence α. Show the following

(a) $\models \beta \rightarrow \alpha$ implies $\models \beta \rightarrow (\forall x)\alpha^c_x$

(b) $\models \alpha \rightarrow \beta$ implies $\models (\exists x)\alpha^c_x \rightarrow \beta$

(c) If x does not occur free in γ, $\models \big((\forall x)(P(x) \leftrightarrow \gamma\big) \rightarrow \big((\forall x)P(x) \vee (\forall x)\neg P(x)\big)$.

(d) If x does not occur free in γ, $\vdash \big((\forall x)(P(x) \leftrightarrow \gamma\big) \rightarrow \big((\forall x)P(x) \vee (\forall x)\neg P(x)\big)$. (That is, provide a tableaux proof.)

Answer. (a) and (b) was from the last homework assignment. (c) and (d) contained a typo, the antecedent was supposed to be $(\forall x)(P(x) \leftrightarrow \gamma$, not $(\forall x)(P(x) \rightarrow \gamma$. Everyone got full credit for this.

(a). This is problem is equivalent to a simple case on the last homework:

$$\beta \models \alpha \quad \text{implies} \quad \beta \models (\forall x)\alpha^c_x.$$

Suppose $\models \beta \rightarrow \alpha$ and let $\mathcal{A}$ be any structure. Suppose that $\beta \rightarrow (\forall x)\alpha$ is not true. Then $v_{\mathcal{A}}(\beta) = \mathbf{T}$ but $v_{\mathcal{A}}((\forall x)\alpha^c_x) = \mathbf{F}$. Let $a \in A$ be such that $v_{\mathcal{A}}([\alpha^c_x]^x_a) = \mathbf{F}$. $\mathcal{A}$ already interprets c, so let $\mathcal{A}^*$ be the structure just like $\mathcal{A}$, except $c^{\mathcal{A}} = a$. Then $v_{\mathcal{A}^*}(\alpha) = \mathbf{F}$. But $v_{\mathcal{A}^*}(\beta) = v_{\mathcal{A}}(\beta)$, since c does not occur in β. Thus, $v_{\mathcal{A}^*}(\beta \rightarrow \alpha) = \mathbf{F}$, which contradicts our hypothesis.

Alternatively, consider any structure $\mathcal{A}$ in which $v_{\mathcal{A}}(\beta) = \mathbf{T}$, so that $v_{\mathcal{A}}(\alpha) = \mathbf{T}$. Fix $a \in A$ and let $\mathcal{A}^*$ be the structure just like $\mathcal{A}$, except $c^{\mathcal{A}} = a$. Then $v_{\mathcal{A}^*}(\beta) = \mathbf{T}$, since c does not occur in β, so $v_{\mathcal{A}^*}(\alpha) = \mathbf{T}$. Since $a \in A$ was arbitrary, $v_{\mathcal{A}}((\forall x)\alpha^c_x) = \mathbf{T}$.

(b). Suppose $\models \alpha \rightarrow \beta$ and show that $\models (\exists x)\alpha^c_x \rightarrow \beta$. Let $\mathcal{A}$ be any structure in which $(\exists x)\alpha^c_x\alpha$ is true, and let $a \in A$ such that $v_{\mathcal{A}}([\alpha^c_x]^x_a) = \mathbf{T}$. Let $\mathcal{A}^*$ be the structure just like $\mathcal{A}$ except that $c^{\mathcal{A}} = a$. Then $v_{\mathcal{A}^*}(\alpha) = \mathbf{T}$, so that $v_{\mathcal{A}^*}(\beta) = \mathbf{T}$. Since c does not occur in β, $v_{\mathcal{A}}(\beta) = \mathbf{T}$.

(c),(d) Homework 6

4. Semantic Tableaux Equivalence

If α and β are first-order sentences, then α and β are *tableaux equivalent* if both conditionals are provable

$$\vdash \alpha \rightarrow \beta \qquad \text{and} \qquad \vdash \beta \rightarrow \alpha$$

The following pairs of sentences are not tableaux equivalent. For each pair of sentences, show which conditionals are tableaux provable and provide a counterexample for those that are not

$$\begin{array}{ll} (a) & (\exists x)\big(P(x) \rightarrow Q(x)\big), \quad (\exists x)\big(P(x) \wedge Q(x)\big) \\ (b) & (\forall x)\big(P(x) \rightarrow Q(x))\big), \quad (\forall x)\big(P(x) \wedge Q(x)\big) \end{array}$$

Consider the structure $A = \{1\}$, and $P^{\mathcal{A}} = Q^{\mathcal{A}} = \emptyset$. In this structure:

- $(\exists x)\big(P(x) \rightarrow Q(x))\big)$ is true and $(\exists x)\big(P(x) \wedge Q(x))\big)$ is false.
- $(\forall x)\big(P(x) \rightarrow Q(x))\big)$ is true and $(\forall x)\big(P(x) \wedge Q(x))\big)$ is false.

The other implications are tableaux provable:

$$\begin{array}{c} (1)\ (\exists x)\big(P(x) \wedge Q(x)\big) \\ (2)\ \neg\,(\exists x)\big(P(x) \rightarrow Q(x)\big) \\ (3)\ \big(P(a) \wedge Q(a)\big)\ (1) \\ (4)\ P(a)\ (3) \\ (5)\ Q(a)\ (3) \\ (6)\ \neg\,\big(P(a) \rightarrow Q(a)\big)\ (2) \\ (7)\ P(a)\ (6) \\ (7)\ \neg\,Q(a)\ (6) \\ \otimes\ (5,7) \end{array}$$

$$\begin{array}{c} (1)\ (\forall x)\big(P(x) \wedge Q(x)\big) \\ (2)\ \neg\,(\forall x)\big(P(x) \rightarrow Q(x)\big) \\ (3)\ \neg\,\big(P(a) \rightarrow Q(a)\big)\ (2) \\ (4)\ P(a)\ (3) \\ (5)\ \neg\,Q(a)\ (3) \\ (6)\ \big(P(a) \wedge Q(a)\big)\ (1) \\ (7)\ P(a)\ (6) \\ (7)\ Q(a)\ (6) \\ \otimes\ (5,7) \end{array}$$

5. Constructing Models

Let

$$\begin{array}{rl} \text{(trans)} & (\forall x)(\forall y)(\forall z)\big((R(x,y) \wedge R(y,z)) \rightarrow R(x,z)\big) \\ \text{(sym)} & (\forall x)(\forall y)\big((R(x,y) \rightarrow R(y,x)\big) \\ \text{(refl)} & (\forall x)R(x,x) \\ \text{(assym)} & (\forall x)(\forall y)\big(R(x,y) \rightarrow \neg R(y,x)\big) \\ \text{(nontriv)} & (\forall x)(\exists y)R(x,y) \end{array}$$

Use semantic tableaux to show the following

(a) (trans), (sym) $\not\models$ (refl). (You must provide a counterexample).

(b) (trans), (sym), (nontriv) $\vdash$ (refl) (Provide a tableaux proof.)

(c) Is the set of sentences $\{$(assym), (nontriv)$\}$ consistent? That is, $\{$(assym), (nontriv)$\} \vdash \bot$? If so, provide a contradictory tableaux. If not, provide a structure in which both sentences are true.

(a). Let $A = \{1\}$ and $R^{\mathcal{A}} = \emptyset$. Then both (trans) and (sym) are true and (refl) is false.

(b). I allowed some simplification in the tableau by applying the type-C rule to multiple quantifiers at once.

$$(1)\ (\forall x)(\forall y)(\forall z)\big((R(x,y)\ \wedge\ R(y,z))\ \to\ R(x,z)\big)$$

$$(2)\ (\forall x)(\forall y)\big((R(x,y)\ \to\ R(y,x)\big)$$

$$(3)\ (\forall x)(\exists y)R(x,y)$$

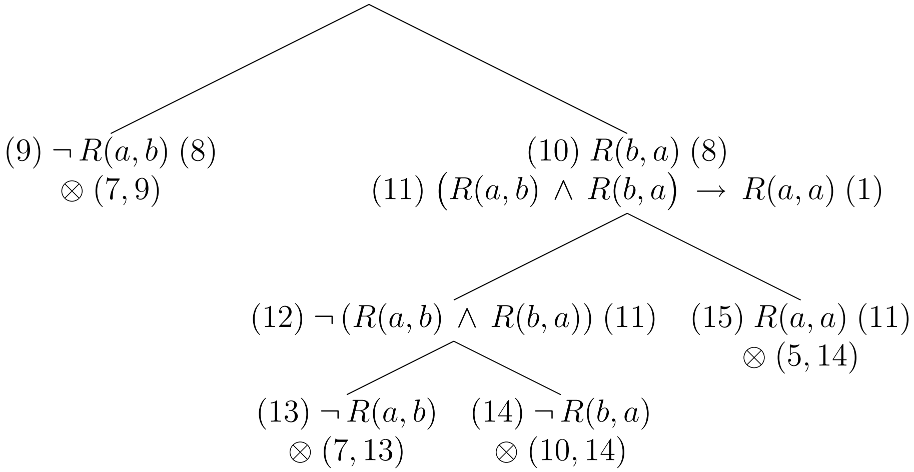
$$(4)\ \neg\,(\forall x)R(x,x)$$

$$(5)\ \neg\,R(a,a)\ (4)$$

$$(6)\ (\exists y)R(a,y)\ (3)$$

$$(7)\ R(a,b)\ (6)$$

$$(8)\ R(a,b)\ \to\ R(b,a)\ (2)\ x \mapsto a, y \mapsto b$$



(c). The simplest structure satisfying both (assym) and (nontriv) has three elements $A = \{1,2,3\}$ and $R^{\mathcal{A}} = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle\}$.

6. Boolean and First-Order Valuation

Let Σ be a set of first-order *quantifier-free* sentences whose constants all come from the set $A = \{a_0, a_1, \ldots\}$.

> Prove the following: If there is a Boolean assignment in which all the sentences of Σ are true, then there is a first-order assignment in which all the sentences of Σ are true.

Hint. Start with a Boolean assignment in which all the sentences of Σ are true, and use this to produce a first-order structure. Then prove that every sentence of Σ is true in this structure by structural induction.

Homework 6