Math481 | Midterm 1 | Friday, October 23

Each part of the exam was worth 10 points, for a total of 70 points.

1 Problem

We defined the biconditional as

$$\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha).$$

This allows us to treat the biconditional using unified notation: $\alpha \leftrightarrow \beta$ is a type-A proposition with components $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$, and $\neg(\alpha \leftrightarrow \beta)$ is a type-B proposition with components $\neg(\alpha \rightarrow \beta)$ and $\neg(\beta \rightarrow \alpha)$.

(a) Formulate rules for semantic tableaux for the biconditional using unified notation.

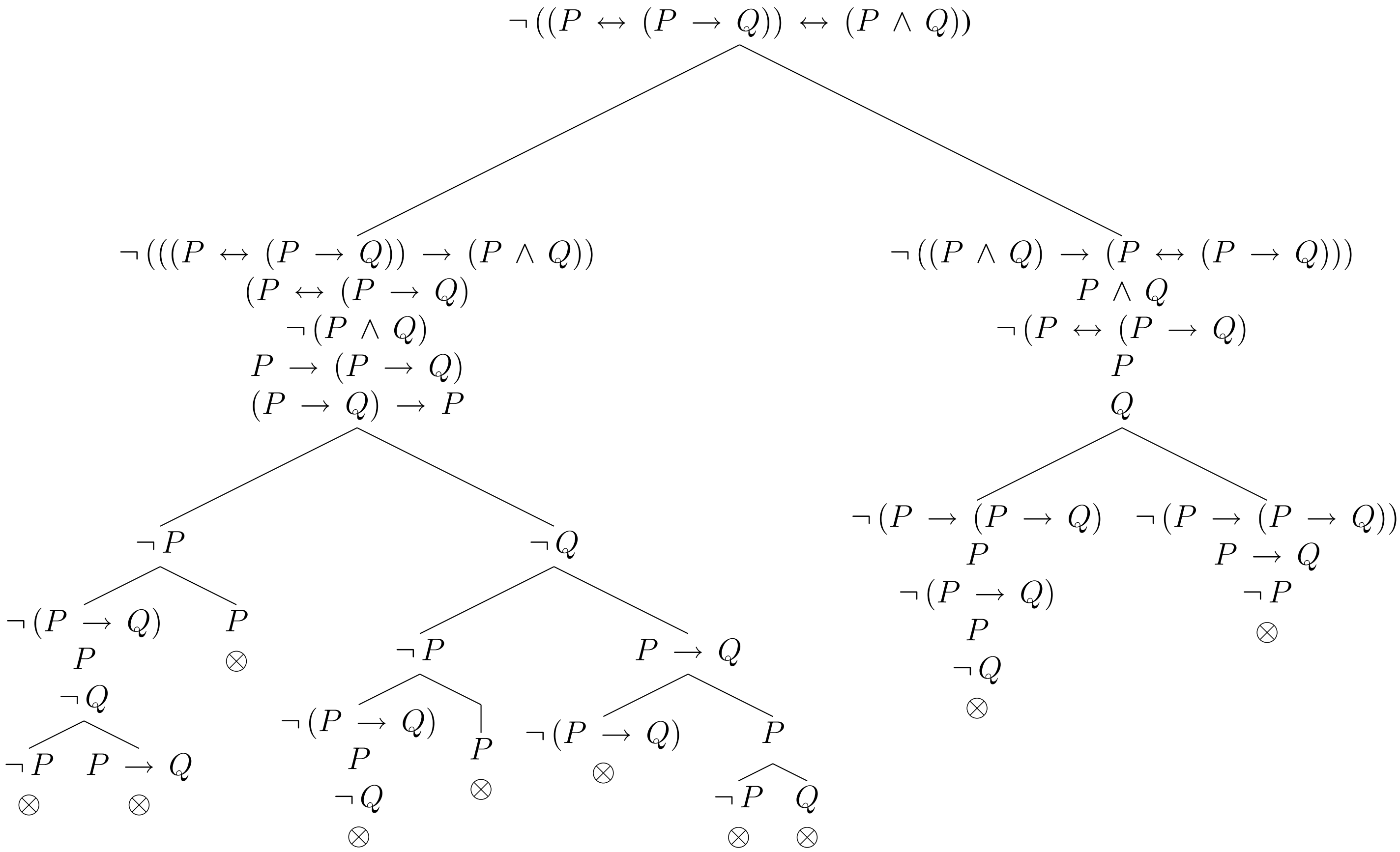
(b) Show $\vdash (P \leftrightarrow (P \rightarrow Q)) \leftrightarrow (P \wedge Q)$ using your rules.

(a). The correct rules used unified notation:

$$\begin{array}{c} \alpha \leftrightarrow \beta \\ \alpha \rightarrow \beta \\ \beta \rightarrow \alpha \end{array} \qquad\qquad \begin{array}{c} \neg(\alpha \leftrightarrow \beta) \\ \swarrow \quad \searrow \\ \neg(\alpha \rightarrow \beta) \quad \neg(\beta \rightarrow \alpha) \end{array}$$

It was acceptable to take these rules another step and apply tableaux rules to the conditionals and negated conditionals.

(b). I was looking for correct application of the tableaux rules for the conditional, so you could get full credit here while missing (a).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2 Problem

Prove the following using the derived rules for natural deduction given on the handout.

$$(a) \qquad \vdash_{\text{nd}} (P \wedge (Q \vee R) \rightarrow ((P \wedge Q) \vee (P \wedge R))$$
$$(b) \qquad \vdash_{\text{nd}} (\neg P \vee \neg Q) \rightarrow \neg (P \wedge Q)$$

I accepted proofs based on the natural deduction rules and the unified rules. It was not acceptable to quote a tautology in a proof without giving a proof. Both proofs are entirely straightforward, and you can use the propositional connectives to drive the proof.

(a). The key to this proof is the correct application of $\vee$-elimination. A common mistake was that both disjunctions must derive *the same proposition* to use $\vee$-elimination.

1	$P \wedge (Q \vee R)$	
2	P	$\wedge$E, 1
3	$Q \vee R$	$\wedge$E, 1
4	Q	
5	$P \wedge Q$	$\wedge$I, 2, 4
6	$(P \wedge Q) \vee (P \wedge R)$	$\vee$I, 5
7	R	
8	$P \wedge R$	$\wedge$I, 2, 7
9	$(P \wedge Q) \vee (P \wedge R)$	$\vee$I, 8
10	$(P \wedge Q) \vee (P \wedge R)$	$\vee$E, 3, 4–6, 7–9
11	$(P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))$	$\rightarrow$I, 1–10

(b). The variation here was to use the $\neg$-introduction rule, together with $\vee$-elimination.

1	$\neg P \vee \neg Q$	
2	$\quad P \wedge Q$	
3	$\quad\quad \neg P$	
4	$\quad\quad P$	$\wedge$**E**, 2
5	$\quad\quad \bot$	$\bot$**I**, 3, 4
6	$\quad\quad \neg Q$	
7	$\quad\quad Q$	$\wedge$**E**, 2
8	$\quad\quad \bot$	$\bot$**I**, 6, 7
9	$\quad \bot$	$\vee$**E**, 1, 3–5, 6–8
10	$\neg (P \wedge Q)$	$\neg$**I**, 2–9
11	$(\neg P \vee \neg Q) \rightarrow \neg (P \wedge Q)$	$\rightarrow$**I**, 1–10

3 Problem

Prove the following directly (do not appeal to Soundness and Completeness):

$$\text{If } \Gamma, \alpha \vdash_{\text{nd}} \beta \ \text{ and } \ \Gamma, \neg\,\alpha \vdash_{\text{nd}} \beta, \ \text{ then } \ \Gamma \vdash_{\text{nd}} \beta.$$

You may assume that any tautology on the list has a proof. (You do not need to provide a proof.)

This is a special case of a problem I put on the study guide:

$$\text{If } \Gamma, \alpha \vdash_{\text{nd}} \beta \ \text{ and } \ \Gamma, \neg\,\alpha \vdash_{\text{nd}} \beta, \ \text{ then } \ \Gamma, (\alpha \vee \neg\,\alpha) \vdash_{\text{nd}} \beta.$$

Since $(\alpha \vee \neg\,\alpha)$ is a tautology, so provable in the natural deduction system, you can always stick in a proof whenever you appeal to it.

Let τ_1 be a proof of $(\alpha \vee \neg\,\alpha)$, τ_2 a proof for $\Gamma, \alpha \vdash_{\text{nd}} \beta$ and τ_3 a proof for $\Gamma, \neg\,\alpha \vdash_{\text{nd}} \beta$. Then the following is a proof for $\Gamma \vdash_{\text{nd}} \beta$:

$$\begin{array}{l|l|l}
\vdots & \tau_1 & \\
n & \alpha \vee \neg\,\alpha & \\
n+1 & \quad \mid\ \alpha & \\
\vdots & \quad \mid\ \tau_2 & \\
m & \quad \mid\ \beta & \\
m+1 & \quad \mid\ \neg\,\alpha & \\
\vdots & \quad \mid\ \tau_3 & \\
p & \quad \mid\ \beta & \\
p+1 & \beta & \vee\text{E}, n, n\text{–}m, m+1\text{–}p
\end{array}$$

4 Problem

Extend the rules of semantic tableaux to include the *splitting rule*:

$$\alpha \qquad \neg\,\alpha$$

Call tableaux that use the splitting rule *synthetic tableaux*. Let $\Gamma \vdash_{\mathrm{ST}} \alpha$ if there is a synthetic tableau proof of α from premises Γ.

Show the following inference can be derived using synthetic tableaux (do not appeal to the soundness and completeness theorem):

$$\text{If } \Gamma \vdash_{\mathrm{ST}} \alpha \ \text{ and } \ \Gamma, \alpha \vdash_{\mathrm{ST}} \beta, \ \text{ then } \ \Gamma \vdash_{\mathrm{ST}} \beta.$$

Let τ_1 be a proof for $\Gamma \vdash_{\mathrm{ST}} \alpha$ and τ_2 be a proof for $\Gamma, \alpha \vdash_{\mathrm{ST}} \beta$. Notice that τ_1 begins with $\neg\,\alpha$ at the root. The following is a synthetic tableau proof for $\Gamma \vdash_{\mathrm{ST}} \beta$.

$$\neg\,\beta$$

$$\alpha \qquad \neg\,\alpha$$

$$\tau_2 \qquad \tau_1$$

$$\otimes \qquad \otimes$$

τ_1 is a synthetic proof which uses only premises from Γ, and although τ_2 may introduce α, we can eliminate all these instances of α, and use the fact that α at the top of the left branch is accessible from every node in τ_2. So, this is a synthetic tableau proof of β which only uses premises from Γ.

5 Problem

If $\alpha \rightarrow \beta$ is a tautology and α and β share no propositional symbols, then one of $\neg\alpha$ or β must also be a tautology.

Hint. Assume neither $\neg\alpha$ nor β is a tautology and derive a contradiction.

Suppose neither $\neg\alpha$ nor β is a tautology. Let v_1 be a valuation for which $v_1(\neg\alpha) = \mathbf{F}$ and v_2 a valuation for which $v_2(\beta) = \mathbf{F}$. Let v be a valuation satisfying $v(P) = v_1(P)$ when P is a propositional symbol occurring in $\neg\beta$ and $v(P) = v_2(P)$ otherwise. Since $\neg\alpha$ and β share no propositional symbols,

$$v(\neg\alpha) = v_1(\neg\alpha) = \mathbf{F} \quad \text{and} \quad v(\beta) = v_1(\beta) = \mathbf{F}.$$

Since $v(\alpha) = \mathbf{T}$ and $v(\beta) = \mathbf{F}$, it follows that $v(\alpha \rightarrow \beta) = \mathbf{F}$, so is not a tautology.

It is essential that you justify the reason that there is single valuation which falsifies both $\neg\alpha$ and β.

6 Problem

Prove the following by Structural Induction. No proposition built only from the propositional symbols and $\{\land, \lor\}$ can be a tautology.

Hint. Look at the truth table for for $\land$ and $\lor$.

Both $\alpha \land \beta$ and $\alpha \lor \beta$ are false when both propositions α and β are false. Let $v(P) = \mathbf{F}$ for all propositional symbols. Then $v(\gamma) = \mathbf{F}$ for each proposition built from the propositional symbols together with $\{\land, \lor\}$. The proof is by Structural Induction on γ.

For the basis step, $v(P) = \mathbf{F}$ for all propositional symbols. Since $\bot$ and $\top$ do not in the propositions we are considering, these cases can be ignored.

For the inductive step, we need only concern ourselves with the connectives $\land$ and $\lor$. Suppose $v(\alpha) = \mathbf{F} = v(\beta)$. The $v(\alpha \land \beta) = \mathbf{F}$ and $v(\alpha \lor \beta) = \mathbf{F}$.

Therefore, no proposition build only from propositional symbols and the connectives $\{\land, \lor\}$ can be a tautology.

7 Problem

Read. If you have written out a proof of Part (c) of the Compactness Theorem from Homework 3, and are satisfied with it, you may submit it as your answer to this question. You will not need to re-write it here.

Definition. A set Γ of propositions (possibly infinite) is *finitely satisfiable* if every finite subset Γ_0 is satisfiable.

Compactness Theorem. Every finitely satisfiable set of propositions is satisfiable.

Question. Let Γ be a set of propositions such that for any interpretation v there is at least one proposition in Γ true under v. Show that there is a finite subset $\Gamma_0 = \gamma_1, \ldots, \gamma_n$ of Γ such that $\gamma_1 \lor \ldots \lor \gamma_n$ is a tautology.