1 Instructions for the Final Exam

You have the option of doing basic questions or advanced questions. There are three parts to the exam: A,B,C (Basic and Advanced), and you must do either the basic or advanced part for each of A,B,C. For example, you may choose to do basic parts A and B and the advanced part of C.

2 Basic Questions

A. Craig Interpolation

There are two parts to this question.

Find interpolants using a contradictory biased tableau as in Lecture 33.

1a. $(A \to C) \to ((B \to D) \to ((A \vee B) \to (C \vee D)))$

1b. $((\forall x)(P(x) \to \neg Q(x)) \wedge P(c)) \to \neg Q(c)$.

1c. $((R(a) \to (\exists x)P(x)) \to Q(b)) \to ((\forall x)(S(c) \wedge P(x)) \to (S(c) \wedge Q(b)))$.

The propositional version of Craig's Interpolation Theorem has a simpler proof. In the following questions you will work in propositional logic. The relation $\Gamma \models_{\text{taut}} \alpha$ means that Γ tautologically implies α.

For any proposition α and propositional symbol A, let $\alpha^A_\top$ be the proposition obtained from α by replacing A by $\top$. Similarly for $\alpha^A_\bot$. Then let $\alpha^A_* = \alpha^A_\top \vee \alpha^A_\bot$. Show that

2a. $\alpha \models_{\text{taut}} \alpha^A_*$.

2b. If $\alpha \models_{\text{taut}} \beta$ and A does not appear in β, then $\alpha^A_* \models_{\text{taut}} \beta$.

2c. (Interpolation Theorem) If $\alpha \models_{\text{taut}} \beta$, then there is some γ all of whose sentence symbols occur in both α and β and such that $\alpha \models_{\text{taut}} \gamma$ and $\gamma \models_{\text{taut}} \beta$.

B. Symmetric Gentzen Relations

Consider a binary relation $\Rightarrow$ between finite sets – we shall write $S \Rightarrow T$ to mean that S stands in the relation $\Rightarrow$ to T. We will also use the convention that when S is a set and X a sentence S, X is the set $S \cup \{X\}$.

Define $\Rightarrow$ to be a *symmetric Gentzen relation* if the following conditions hold for every pair of finite sets (possibly empty) S and T.

C_0 For any sentence X

$$S, X \Rightarrow T, X$$
$$S, X, \neg X \Rightarrow T$$
$$S \Rightarrow T, X, \neg X$$

C_1 For any type-A sentence α with components α_1, α_2

(a) If $S, \alpha_i \Rightarrow T$, then $S, \alpha \Rightarrow T$ (for $i = 1, 2$).

(b) If $S \Rightarrow T, \alpha_1$ and $S \Rightarrow T, \alpha_2$, then $S \Rightarrow T, \alpha$.

C_2 For any type-B sentence β with components β_1, β_2

(a) If $S, \beta_1 \Rightarrow T$ and $S, \beta_2 \Rightarrow T$, then $S, \beta \Rightarrow T$.

(b) If $S \Rightarrow T, \beta_i$, then $S \Rightarrow T, \beta$ (for $i = 1, 2$).

C_3 For any type-C sentence γ

(a) If $S, \gamma(t) \Rightarrow T$, then $S, \gamma \Rightarrow T$.

(b) If $S \Rightarrow T, \gamma(a)$, then $S \Rightarrow T, \gamma$, provided a is a parameter that does not occur in any of S, T, γ.

C_4 For any type-D sentence δ

(a) If $S, \delta(a) \Rightarrow T$, then $S, \delta \Rightarrow T$, provided a is a parameter that does not occur in any of S, T, δ.

(b) If $S \Rightarrow T, \delta(t)$, then $S \Rightarrow T, \delta$

Now define $S \Rightarrow_0 T$ to mean that every interpretation which satisfies *all* the sentences of S also satisfies *at least one* sentence of T.

1. (Basic) Prove that $\Rightarrow_0$ is a symmetric Gentzen relation.

2. (Advanced) Prove the following: For any symmetric Gentzen relation $\Rightarrow$ and for any finite sets S, T if $S \Rightarrow_0 T$ then $S \Rightarrow T$.

   Hint. Define the conjugate $\overline{X}$ of a sentence X as follows: let $\overline{X} = Y$ if $X = \neg Y$ for some sentence Y and $\overline{X} = \neg Y$ if $X = Y$ and Y is not a negation. Define a property $\mathcal{I}$ on finite sets U as follows: $U \in \mathcal{I}$ if for any way of assigning biases L and R to the sentences of U so that if S is the left-biased sets and T is the conjugate of the right-biased sets:

   $$S = \{X : X \text{ is left-biased}\} \qquad T = \{\overline{Y} : Y \text{ is right-biased}\}$$

   then $S \Rightarrow T$. Show that if $S \Rightarrow T$ is a symmetric Gentzen relation if and only if $\mathcal{I}$ is a first-order inconsistency property. Use this to prove 2.

C. Maximal Consistency and Completeness

A set Γ of sentences is said to be *consistent* if $\Gamma \not\models \bot$. A set Γ of sentences is *maximally consistent* if it is consistent and there is no consistent set of sentences Σ *properly* extending Γ, that is, $\Gamma \subsetneq \Sigma$. A set of sentences is *complete* if for every sentence ϕ either

$$\Gamma \models \phi \quad \text{or} \quad \Gamma \models \neg\phi.$$

Prove the following statements are equivalent.

1. The set of logical consequences of Γ, $\{\phi : \Gamma \models \phi\}$, is a maximal consistent set.

2. Γ is complete.

3. If $\mathcal{A}$ and $\mathcal{B}$ are two structures in which all the sentences of Γ are true, then $\mathcal{A}$ and $\mathcal{B}$ are *elementary equivalent*. That is, $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences: $\mathcal{A} \models \phi$ if and only if $\mathcal{B} \models \phi$, for every sentence ϕ.

3 Advanced Questions

A. Application of Compactness

Let $\mathcal{L}$ be a language with a single relation R. A structure $\mathcal{A}$ for this language is a *linear ordering*, if the following sentences are true in the structure

$$(\forall x)\neg R(x,x)$$
$$(\forall x)(\forall y)(\forall z)((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$$
$$(\forall x)(\forall y)(R(x,y) \vee R(y,x) \vee x = y).$$

You can treat equality, =, as a binary relation which has a fixed interpretation in a structure:

$$=^{\mathcal{A}} \text{ is } \big\{\langle a,a\rangle \,:\, a \in A\big\}.$$

It is common to read the relation $R^{\mathcal{A}}$ in a linear ordering as "less than".

An infinite R-descending sequence in a linear ordering $\mathcal{A}$ is an infinite sequence of elements $a_1, a_2, \ldots, a_n, \ldots$ from the domain A such that $\langle_{n+1}, a_n\rangle$ for each n.

Prove that for any infinite linear ordering $\mathcal{A}$ there is a linear ordering $\mathcal{A}'$ satisfying exactly the same sentences of $\mathcal{L}$ and $\mathcal{A}'$ also has an infinite R-descending sequence.

Hint. Use the Compactness Theorem in the following form

- **Compactness Theorem**. If W is a set of sentences of $\mathcal{L}$ which is finitely satisfiable, then W is satisfiable.

B. Henkin Proof of Completeness

The goal of this problem is to provide an alternative approach to proving the completeness theorem, based on a slick argument due to Raymond Smullyan as suggested by Leon Henkin. We will be working with a first-order language $\mathcal{L}$. For simplicity, assume that $\mathcal{L}$ is a purely relational language – it has no constant terms or function symbols. We will take $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ to be a countably infinite set of parameters and $\mathcal{L}^A$ the language $\mathcal{L}$ extended to include the new constant symbols.

Definition. A set U of sentences is *truth-functionally inconsistent* if it is not satisfiable in any *Boolean assignment*, in the sense of Definition 24.1.4. (A set which is truth-functionally unsatisfiable is also first-order unsatisfiable.) A set U of sentences *tautologically implies* a sentence X if any Boolean assignment which satisfies all the sentences of

U also satisfies X. (See Definition 24.1.6)

A collection $\mathcal{S}$ of sets of sentences in $\mathcal{L}^A$ is a *synthetic consistency property* if it meets the following conditions for every set U of sentences (the missing (B_2) and (B_3) is intentional)

(B_0) $\mathcal{S}$ is of *finite character*: For any set U, $U \in \mathcal{S}$ if and only if for every finite subet $U_0 \subset U$, $U_0 \in \mathcal{S}$.

(B_1) If U is truth-functionally inconsistent, then $U \notin \mathcal{S}$.

(B_4) If $U \cup \{\gamma\} \in \mathcal{S}$ and γ is type-C, then $U \cup \{\gamma(a)\} \in \mathcal{S}$ for every parameter $a \in A$.

(B_5) If $U \cup \{\delta\} \in \mathcal{S}$ and δ is type-D, then $U \cup \{\delta(a)\} \in \mathcal{S}$ for any parameter $a \in A$ which does not occur in U or in δ.

(B_6) If $U \in \mathcal{S}$, then for any sentence X either $U \cup \{X\} \in \mathcal{S}$ or $U \cup \{\neg X\} \in \mathcal{S}$.

If $U \in \mathcal{S}$ we will say that U is $\mathcal{S}$-consistent. Note that the collection of all finitely satisfiable sets is a synthetic consistency property. The collection of all ND-consistent sets (Definition 32.2.1) is a synthetic consistency property. Less obviously, the collection of all tableaux-consistent sets is a synthetic consistency property. (This is not obvious because condition (B_6) cannot be directly shown. If we add the synthetic rule, allowing any branch to split into X and $\neg X$, then we can directly show (B_6) as well.)

Prove the following facts about synthetic consistency properties $\mathcal{S}$.

1. Show (B_7) follows from the other conditions: If U is $\mathcal{S}$-consistent and U tautologically implies X, then $U \cup \{X\}$ is $\mathcal{S}$-consistent.

2. Every synthetic consistency property is a first-order consistency property.

Definition. A set U of sentences is *Henkin complete* if for every type-D sentence $\delta \in U$, there is a parameter $a \in A$ with $\delta(a) \in U$. A set U is *maximally $\mathcal{S}$-consistent* if U is $\mathcal{S}$-consistent and there is no proper superset $V \supsetneq U$ which is $\mathcal{S}$-consistent.

Prove the following for any synthetic consistency property $\mathcal{S}$.

3. If U is maximally $\mathcal{S}$-consistent and Henkin complete, then U is a Hintikka set over the set of parameters A.

4. If U is maximally $\mathcal{S}$-consistent and Henkin complete, then there is a structure $\mathcal{A}$ satisfying the following:

$$V_{\mathcal{A}}(X) = \mathbf{T} \quad \text{if and only if} \quad X \in U.$$

The final step is to show that every $\mathcal{S}$-consistent set of sentences in the language $\mathcal{L}$ can be extended to a maximally $\mathcal{S}$-consistent set in the language $\mathcal{L}^A$ which is also Henkin complete. Let $X_1, X_2, \ldots, X_n, \ldots$ be an enumeration of all sentences of $\mathcal{L}^A$. Let U be a $\mathcal{S}$-consistent set of sentences from the language $\mathcal{L}$. Construct a sequence of sets U_n as follows.

$$U_0 = U$$
$$U_{n+1} = \begin{cases} U_n \cup \{X_{n+1}\} & \text{if this set is } \mathcal{S}\text{-consistent} \\ U_n \cup \{\neg X_{n+1}\} & \text{otherwise.} \end{cases}$$

In addition if $X_{n+1} = \delta$ then add the sentence $\delta \to \delta(a)$ to U_{n+1}, where $a \in A$ is a parameter that does not occur in U_{n+1} or δ. The sentence $\delta \to \delta(a)$ is called a Henkin sentence and the parameter a is called a Henkin witness.

Let $U_\infty = \cup_n U_n$. Prove the following facts about U_∞.

5. Show that U_{n+1} is $\mathcal{S}$-consistent. (Think carefully why it was okay to add the Henkin sentence $\delta \to \delta(a)$.)

6. Show that U_∞ is maximally $\mathcal{S}$-consistent. (We need the finite character of $\mathcal{S}$ to conclude that U_∞ is $\mathcal{S}$-consistent.)

7. Show that U_∞ is Henkin complete.

It now follows by (3), (6), and (7) that every $\mathcal{S}$-consistent set is satisfiable.

C. Symmetric Gentzen Relations

Do question 2 of part B on the Basic Exam. It is not necessary to do part 1 to get credit for part 2.