1 Questions Required for Everyone

The following questions are required for everyone.

Translation

(A). Use the following translation key:

- UD: All mathematical objects.
- $N(x)$: "x is a number",
- $I(x)$: "x is interesting",
- $L(x, y)$: "x is less than y"
- 0: a constant symbol denoting then number zero.

Translate the following English sentences into first-order logic.

a. Zero is less than any number.

b. If any number is interesting, then zero is interesting.

c. No number is less than zero.

d. Any uninteresting number with the property that all smaller numbers are interesting is certainly interesting.

e. There is no number such that all numbers are less than it.

f. There is no number such that no number is less than it.

(B). Let $I(x, y)$ be a two place predicate which will be interpreted as identity in any structure $\mathcal{A}$. That is,

$$I^{\mathcal{A}} = \{\langle a, a\rangle : a \in A\}.$$

Find a sentence of first-order logic which says that the domain has at least one element and a sentence which says that the domain has exactly one element. Find a sentence of first-order logic which says that the domain has at least two elements and a sentence of first-order logic which says that the domain has exactly two elements.

2 Basic Problems

The following questions establish basic skills in first-order logic. You must establish competence in this area, but you can opt to submit solution to the Advanced problems in place of these.

Translation

Use the following translation key:

- UD: All people.
- $T(x, y)$: x can trap y.
- h: denotes Sherlock Holmes
- m: denotes Moriarty.

Translate the following English sentences into first-order logic.

a. Holmes can trap anyone who can trap Moriarty.

b. Homes can trap anyone whom Moriarty can trap.

c. Homes can trap anyone who can be trapped by Moriary.

d. If anyone can trap Moriarty, then Holmes can.

e. If everyone can trap Moriary, then Homes can.

f. Anyone who can trap Holmes can trap Moriarty.

g. No one can trap Holmes unless he can trap Moriarty.

h. Everyone can trap someone who cannot trap Moriarty.

i. Anyone who can trap Holmes can trap anyone whom Moriarty can trap.

Structural Induction

Prove: For an constants a and b and *distinct* variables x and y, $[\phi^x_a]^y_b = [\phi^y_b]^x_a$.

Validity

Show that the following are valid.

a $(\forall x)P(x) \rightarrow P(c)$

b $(\exists x)(P(x) \rightarrow (\forall x)P(x))$

c $(\exists y)(\forall x)R(x, y) \rightarrow (\forall x)(\exists y)R(x, y)$.

d $(\forall x)P(x) \rightarrow \neg(\exists x)\neg P(x)$.

3 Advanced Problems

You can submit these problem in lieu of the Basic problems. Anyone can attempt these without penalty, provided they do the Basic problem set.

Infinite Interpretations (Advanced)

Let α be the conjunction of the following sentences:

$$(\forall x)\neg R(x, x)$$
$$(\forall x)(\forall y)(\forall z)((R(x, y) \wedge R(x, z)) \rightarrow R(x, z))$$
$$(\forall x)(\exists y)R(x, y)$$

Given an infinite model for α and prove that there is no finite model of α.

Interpretations (Advanced)

Fix a language $\mathcal{L}$ without constant symbols. Let A be a set. One sometimes speaks of a sentence being satisfiable or valid in a given universe A. Say that a sentence ϕ is *satisfiable in* A if there is a structure $\mathcal{A}$ whose universe is A and $v_{\mathcal{A}}(\phi) = \mathbf{T}$. A sentence is *valid in* A if it is true in every structure whose universe is A.

(a) Show that if a sentence is satisfiable in a domain A, then it is satisfiable in every domain $B \supseteq A$.

(b) Show that if a sentence is valid in a domain B, then it is valid in every domain $A \subseteq B$.

It is a result due to Löwenheim that if a sentence is satisfiable, then it is satisfiable in a countable domain. This result was extended by Skolem to sets of sentences: If a countable set of sentences is satisfiable, then it is satisfiable in a countable domain. These results have a profound impact on the foundations of mathematics, since they imply that if we have a set of axioms with an intended interpretation whose domain is uncountable, such as axioms which are true over the domain of real numbers, then there is a countable domain in which the axioms are true. Since there is a first-order set of axioms (the Zermelo-Frankel axioms for set theory) which comprises all of mathematics (the intended domain), there is a much smaller domain, one which is only countable, which satisfies the set of axioms.

Compactness (Advanced)

Use the Compactness Theorem for propositional logic in the form: Every finitely satisfiable set is satisfiable.

A *linearly ordered set* $(A, <)$ is a set A together with a binary relation $<$ which satisfies the following conditions:

$$
\begin{array}{l}
(\forall x) x \not< x \\
(\forall x)(\forall y)(\forall z)((x < y \wedge y < z) \rightarrow x < z) \\
(\forall x)(\forall y)(x < y \vee y < x \vee x = y).
\end{array}
$$

$(A, <)$ is a *partially order set* if $<$ satisfies the first two conditions.

A partial order has *width at most* n if every set of pairwise incomparable elements has size at most n. A chain in a partial order $<$ is simply a subset of the order which is linearly ordered by $<$. Prove that an (countable) infinite partial order of width at most 3 can be divided into 3 chains (not necessarily disjoint) if every finite order of width at most 3 can be so divided.

Note. Dilworth's theorem states that any partial order of width at most n can be divided into n chains. So, Dilworth's theorem for infinite orders follows from Dilworth's theorem for finite orders by Compactness. (The choice of 3 in this problem is only to avoid tedious notation; the same argument works for any n.)

Hint. Let the elements of the order be $\{p_n : n \in \mathbb{N}\}$. Consider the propositions $p_i < p_j, A(p_i), B(p_i), C(p_i)$ for $i, j \in \mathbb{N}$. Think of the propositions $A(p_i)$ as saying $p_i \in A$ and similarly for $B(p_i), C(p_i)$. Now write down sets of propositions which express: each of A, B, C is a chain, and the order has width 3.