Math481 Homework 3 due Wednesday, October 21

Natural Deduction

Prove the following propositions using the unified deduction rules from Lecture 9:

$$\begin{array}{ll} (a) & \vdash_{\text{ND}} ((P \to Q) \to P) \to P \\ (b) & \vdash_{\text{ND}} \neg (P \uparrow Q) \to (\neg P \uparrow \neg Q) \end{array}$$

Prove the following propositions using the derived rules from Lecture 10:

$$\begin{array}{ll} (c) & \vdash_{\text{ND}} (\alpha \to \beta) \to (\neg (\beta \vee \gamma) \to \neg (\alpha \vee \gamma)) \\ (d) & \vdash_{\text{ND}} (\neg \alpha \to \neg \beta) \to ((\neg \alpha \to \beta) \to \alpha) \\ (e) & \vdash_{\text{ND}} (\alpha \wedge (\beta \vee \gamma) \to ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \\ (f) & P \vee \neg Q, Q \vee \neg R, R \vee \neg S \vdash_{\text{ND}} S \to P \end{array}$$

Proofs from Premises

(a). Prove the *Cut Rule*:

(Cut Rule): If $\Gamma, \alpha \models \beta$ and $\Gamma \models \alpha$, then $\Gamma \models \beta$.

(b). Prove the *Cut Rule* for semantic tableaux:

If $\Gamma, \alpha \vdash \beta$ and $\Gamma \vdash \alpha$, then $\Gamma \vdash \beta$.

(c). Let $\Gamma = \{P_1 \to P2, P_2 \to P_3, P_3 \to P_4, \ldots\}$. Show $\Gamma \models P_1 \to P_n$ for any n.

(d). Show $\Gamma \vdash P_1 \to P_n$ for any n. (This means that $P_1 \to P_n$ is tableau-derivable from Γ.)

Tautological Implication

Say that a set Γ_1 of propositions is *equivalent* to a set Γ_2 if for any proposition α,

$$\Gamma_1 \models \alpha \text{ if and only if } \Gamma_2 \models \alpha$$

A set of propositions Γ is *independent* if for every α

$$\Gamma - \{\alpha\} \not\models \alpha$$

(a) Prove that every finite set s Γ has a finite independent equivalent subset of propositions.

The next two parts are designed to show that for every (possibly infinite) set of propositions Γ there is an independent equivalent set Σ of propositions.

(b). Let Γ be a set of propositions ordered by $\gamma_1, \gamma_2, \ldots$. Find a sequence of propositions $\Gamma' = \beta_1, \beta_2, \ldots$ which is equivalent to Γ, and such that $\beta_{n+1} \models \beta_n$, but $\beta_n \not\models \beta_{n+1}$.

(c). Let $\Gamma' = \beta_1, \beta_2, \ldots$ be a set of propositions as in part (b). Define $\alpha_1 = \beta_1$ and for every $i \geq 1$, $\alpha_{n+1} = (\beta_n \rightarrow \beta_{n+1}$. Prove that $\Delta = \alpha_1, \alpha_2, \ldots$ is independent and equivalent to Γ', and hence equivalent to Γ in part (b).

(d). The set Δ from (c) is need not a subset of Γ. Show that the set $\Gamma = P_0, P_0 \wedge P_1, P_0 \wedge P_1 \wedge P_2, \ldots$ has no equivalent and independent subset.

Compactness

Definition. A set Γ of propositions (possibly infinite) is *finitely satisfiable* if every finite subset Γ_0 is satisfiable.

(a). Show that the property of being finitely satisfiable is a propositional consistency property. Explain why the following form of the Compactness Theorem follows from (a):
Compactness Theorem. Every finitely satisfiable set of propositions is satisfiable.

(b). Prove that the above version of the Compactness Theorem is equivalent to the one proved in class

$$\Gamma \models \beta \quad \text{implies that for some finite } \Gamma_0 \subseteq \Gamma, \Gamma_0 \models \beta.$$

(c). Let Γ be a set of propositions such that for any interpretation v there is at least one proposition in Γ true under v. Show that there is a finite subset $\Gamma_0 = \gamma_1, \ldots, \gamma_n$ of Γ such that $\gamma_1 \vee \ldots \vee \gamma_n$ is a tautology. (Use the Compactness Theorem as given in (b).)

0.1 Weak König's Lemma

Imagine that you now live in a universal in which everyone is immortal, except for the possibility of being executed. Some are chosen to play the following game (due to the devious Raymond Smullyan), whose stake is their life!

- You have an infinite supply of balls, each bearing a positive integer. There are infinitely many balls bearing n's for each number n. You also have a box with infinite capacity, but it starts out empty. On the first day you may place one ball, bearing any number you choose, into the box. On each succeeding day you must remove one ball, but you may replace it with any finite number of *lower numbered* balls. For example, you may throw out a ball marked 58 and replace it with a billion billion 57's. A ball marked 1 cannot be replaced by other balls, once removed.

You have been chosen to play the game. Is their a strategy to keep the game going forever? Or are you doomed?

Hint. Put the balls on a tree $\mathscr{T}$ (that is, label the tree by the number on the ball). You can build your tree from finite sequences of numbers, $\mathbb{N}^*$. Put the first ball at the root. Let a ball b_2 be an immediate successor of another ball b_1 if ball b_2 was one of the balls that replaced b_1. Try to reduce this problem to Brouwer's Fan Theorem 15.2.4.