Math481 Homework 2 due Wednesday, October 7

Boolean Valuations

(1) Prove or Refute: If $\Gamma \models \alpha$ or $\Gamma \models \beta$, then $\Gamma \models (\alpha \vee \beta)$.

(2) Prove or Refute: If $\Gamma \models (\alpha \vee \beta)$, then $\Gamma \models \alpha$ or $\Gamma \models \beta$.

(3) Prove the Deduction Theorem: $\Gamma, \alpha \models \beta$ if and only if $\Gamma \models \alpha \rightarrow \beta$. .

Truth Functional Completeness

(1). Define disjunction $(P \vee Q)$ using only the conditional $\rightarrow$.

(2). Show that set of connectives $\{\bot, \top, \neg, \leftrightarrow, \oplus\}$ is not truth functionally complete. The truth table for $\oplus$ (exclusive-or) is

P	Q	$P \oplus Q$
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Hint. Show that every proposition α built from these connectives and propositional symbols P_0 and P_1 has an even number of **T**'s among the four possible values of $v(\alpha)$.

Variable Liar

There is a cluster of Islands in which, on each island, the lying and truth-telling habits of the people vary from day-to-day. However, on any given day, an inhabitant will tell the truth all day or lie all day.

A *Boolean Island* is an island in which the following three laws hold.

N For any inhabitant A there is an inhabitant who tells the truth on all and only those days in which A lies.

C For any pair of inhabitants A and B there is an inhabitant C who tells the truth on all and only those days on which both A and B tell the truth.

D For any pair of inhabitants A and B there is an inhabitant C who tells lies on all and only those days on which both A and B lie.

1. There is an island called Irving's Island in which the law (**N**) above holds as well as the following additional law:

   I For any pair of inhabitants A and B there is an inhabitant C who lies on all and only those days on which both A tells the truth and B lies.

   Show that Irving's Island is a Boolean Island.

2. Jacob's Island satisfies a single law

   J For any inhabitant A and B, there is an inhabitant C who tells the truth on all and only those days in which both A and B lie.

   Show that Jacob's Island is a Boolean island.

Normal Form

Write each of the following propositions in conjunctive and disjunctive normal form.

$$(1) \qquad (P \to R) \to ((Q \to S) \to ((P \vee Q) \to R)$$
$$(2) \qquad (P \to Q) \to ((Q \to \neg R) \to \neg P)$$

You may use the method of Lemma 7.3.6 and Example 7.3.7 of Lecture 7.

Tableau Proofs

Prove the following tautologies using the semantic tableaux proofs.

$$(1) \qquad (P \to Q) \to ((P \to (Q \to R)) \to (P \to R))$$
$$(2) \qquad ((P \to Q) \wedge (\neg P \to Q)) \to Q$$
$$(3) \qquad ((P \to Q) \to P) \to P$$
$$(4) \qquad ((P \to Q) \wedge (Q \to R)) \to \neg(\neg R \wedge P)$$
$$(5) \qquad (P \leftrightarrow (P \to Q)) \leftrightarrow (P \wedge Q)$$