Boolean Valuations (15 points)

Prove or Refute: If $\Gamma \models \alpha$ or $\Gamma \models \beta$, then $\Gamma \models (\alpha \vee \beta)$. (1). **True**. Suppose $\Gamma \models \alpha$. Then for any valuation v which satisfies each proposition in Γ, $v(\alpha) = \mathbf{T}$, so that $v(\alpha \vee \beta) = \mathbf{T}$. Thus, $\Gamma \models (\alpha \vee \beta)$. A similar argument show that if $\Gamma \models \beta$, then $\Gamma \models (\alpha \vee \beta)$. Since $\Gamma \models \alpha$ or $\Gamma \models \beta$, it follows that $\Gamma \models (\alpha \vee \beta)$. Notice that the argument is *proof by cases*.

Prove or Refute: If $\Gamma \models (\alpha \vee \beta)$, then $\Gamma \models \alpha$ or $\Gamma \models \beta$. (2). **False**. Since $(P \vee \neg P)$ is a tautology, $\Gamma \models (P \vee \neg P)$ for any set of propositions Γ. So, $Q \models (P \vee \neg P)$. However, $Q \not\models P$, let $v(Q) = \mathbf{T}$ and $v(P) = \mathbf{F}$, and $Q \not\models \neg P$, let $v(Q) = \mathbf{T}$ and $v(P) = \mathbf{T}$.

Prove the Deduction Theorem: $\Gamma, \alpha \models \beta$ if and only if $\Gamma \models \alpha \rightarrow \beta$. (3). Suppose $\Gamma, \alpha \models \beta$ and let v be any assignment which satisfies Γ. If $v(\alpha) = \mathbf{F}$, then $v(\alpha \rightarrow \beta) = \mathbf{T}$. If $v(\alpha) = \mathbf{T}$, then $v(\beta) = \mathbf{T}$ by our supposition, so $v(\alpha \rightarrow \beta) = \mathbf{T}$. It follows that $\Gamma \models (\alpha \rightarrow \beta)$. (Another proof by cases. What is the true disjunction here?)

Suppose $\Gamma \models (\alpha \rightarrow \beta)$ and let v be any assignment which satisfies $\Gamma \cup \{\alpha\}$. Then $v(\alpha) = \mathbf{T}$ and $v(\alpha \rightarrow \beta) = \mathbf{T}$, by supposition, so $v(\beta) = \mathbf{T}$ by the truth conditions for the conditional. So, $\Gamma, \alpha \models \beta$.

Truth Functional Completeness (10 points

(1). Define disjunction $(P \vee Q)$ using only the conditional $\rightarrow$.

$P \vee Q \simeq (Q \rightarrow P) \rightarrow P$. You can verify this using the fact that $P \rightarrow Q \simeq \neg P \vee Q$:

$$(Q \rightarrow P) \rightarrow P \simeq \neg(Q \rightarrow P) \vee P \simeq (Q \wedge \neg P) \vee P \simeq P \vee Q$$

(2). Show that set of connectives $\{\bot, \top, \neg, \leftrightarrow, \oplus\}$ is not truth functionally complete. The truth table for $\oplus$ (exclusive-or) is

P	Q	$P \oplus Q$
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

The key idea is to prove by structural induction that for any proposition α built from two propositional symbols P and Q and the connectives $\{\bot, \top, \neg, \leftrightarrow, \oplus\}$, the truth table for α has an even number of **T** in the four entries. Since there are truth tables for two propositional symbols with an odd number of lines with **T**, such as for disjunction, $P \vee Q$, the connectives cannot be truth functionally complete.

For the basis case, $\alpha = \top$ has four **T**'s, $\alpha = \bot$ has 0 **T**'s and both $\alpha = P$ and $\alpha = Q$ have two **T**'s.

Now, suppose α and β have an even number of **T**'s in their truth table, say k and ℓ, respectively, are the number of such rows. Then $\neg\alpha$ will have $4 - k$ rows with **T**, and this will be even since k is even. (Either 0, 2 or 4 rows with **T**.)

For the two binary connectives we will show that α and β must have the same value on an even number of rows. This is clear if one of the propositions has 0 or 4 **T**'s, that is either all rows are **T** or all rows are **F** (since the other propositions has an even number of **T**'s and **F**'s as well). Suppose both propositions have two rows with **T** and two rows with **F**. There are 0, 1 or 2 rows where both have **T**. If there are 0 rows where both have **T**, then they agree on 0 rows; if there are 2 rows where both have **T**, then they agree on all 4 rows. If there is 1 row where they both have **T**, then there must be 1 row where they have **F**, since there are three rows where at least one proposition has a **T**. Now, $\alpha \leftrightarrow \beta$ is **T** on all rows where the truth values agree, which is even, and $\alpha \oplus \beta$ is **T** on all rows where the truth values disagree, which must also be even.

Variable Liar (10 points

There is a cluster of Islands in which, on each island, the lying and truth-telling habits of the people vary from day-to-day. However, on any given day, an inhabitant will tell the truth all day or lie all day.

A *Boolean Island* is an island in which the following three laws hold.

N For any inhabitant A there is an inhabitant who tells the truth on all and only those days in which A lies.

C For any pair of inhabitants A and B there is an inhabitant C who tells the truth on all and only those days on which both A and B tell the truth.

D For any pair of inhabitants A and B there is an inhabitant C who tells lies on all and only those days on which both A and B lie.

1. There is an island called Irving's Island in which the law (**N**) above holds as well as the following additional law:

   I For any pair of inhabitants A and B there is an inhabitant C who lies on all and only those days on which both A tells the truth and B lies.

   Show that Irving's Island is a Boolean Island.

   Answer. It is enough to show that (**D**) holds (or (**C**) holds), since in the presence of (**N**), these two conditions are equivalent. For any inhabitant A, let A' be the inhabitant who lies exactly when A tells the truth (by (**N**)). Let A and B be any inhabitant then by (**I**) there is an inhabitant D who lies on exactly those days when A' tells the truth (that is, A lies) and B lies. So, D fulfills condition (**D**) for A and B.

   Alternatively, let C be an inhabitant who lies on exactly those days in which both A tells the truth and B' lies (that is, B tells the truth). Then C' is one who tells the truth exactly when A and B tell tell the truth, and thus fulfills condition (**C**) for A and B.

2. Jacob's Island satisfies a single law

   J For any inhabitant A and B, there is an inhabitant C who tells the truth on all and only those days in which both A and B lie.

   Show that Jacob's Island is a Boolean island.

Answer. It is enough to show (**N**) holds and any one of (**C**), (**D**), or (**I**). For any inhabiant A, let A' be an inhabitant who tells the truth on exactly those days that A (and A again) lies. So, A' satisfies (**N**). Fix any inhabitants, A and B, and let C be an inhabitant who tells the truth exactly on those days in which both A and B lie. Then C' lies exactly on those days in which both A and B lie. This establishes (**D**).

Normal Form (10 points)

Write each of the following propositions in conjunctive and disjunctive normal form.

$$\begin{array}{ll} (1) & (P \to R) \to ((Q \to S) \to ((P \vee Q) \to R)) \\ (2) & (P \to Q) \to ((Q \to \neg R) \to \neg P) \end{array}$$

You may use the method of Lemma 7.3.6 and Example 7.3.7 of Lecture 7.

The method of constructing the proposition from truth tables is long. Instead, I will reduce into these forms using the De Morgan, negation laws and equivalences from Lecture 5.

(1). The reduction to disjunctive normal form is in two steps:

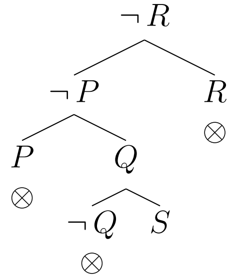
$$\begin{array}{ll} (a) & \neg(P \to R) \vee \big(\neg(Q \to S) \vee (\neg(P \vee Q) \vee R)\big) \\ (b) & (P \wedge \neg R) \vee (Q \wedge \neg S) \vee (\neg P \wedge \neg Q) \vee R. \end{array}$$

We could reduce (b) to conjunctive normal form by using the distribution law, The reduction to conjunctive normal form is by the distribution law:

$$(P \wedge Q) \vee R \simeq (P \vee R) \wedge (Q \vee R),$$

to push disjunctions inside conjunctions from (b) above. However, this produces a very long and complicated proposition. Instead, consider what is required to falsify the proposition: what must a valuation do to make this proposition false. Use a semantic tableau:

$$\begin{array}{c} \neg((P \to R) \to ((Q \to S) \to ((P \vee Q) \to R))) \\ P \to R \\ \neg((Q \to S) \to ((P \vee Q) \to R))) \\ Q \to S \\ \neg((P \vee Q) \to R) \\ P \vee Q \\ \neg R \end{array}$$



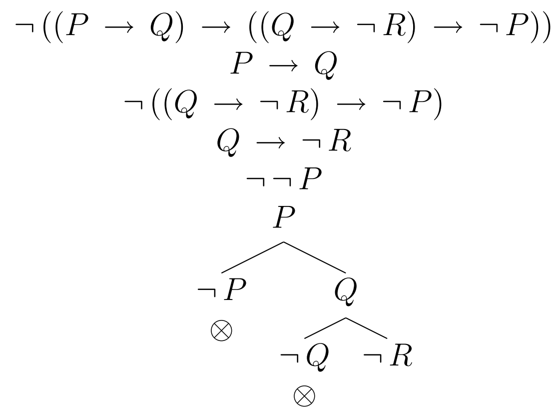
The only way to falsify this proposition is for an assignment $\mathcal{A}$ to do the following: $\mathcal{A}(Q) = \mathcal{A}(S) = \mathbf{T}$ and $\mathcal{A}(P) = \mathcal{A}(R) = \mathbf{F}$. Conjunctive normal form rules this possibility out:

$$\neg Q \lor \neg S \lor P \lor R.$$

(2). The reduction to disjunctive normal form is in two steps:

$$\begin{array}{ll} (a) & \neg(P \to Q) \lor (\neg(Q \to \neg R) \lor \neg P) \\ (b) & (P \land \neg Q) \lor (Q \land R) \lor \neg P \end{array}$$

We could convert (b) using the distribution law, but I will again use semantic tableaux to look for a counterexample.



The only way to falsify this proposition is for an assignment $\mathcal{A}$ to do the following: $\mathcal{A}(P) = \mathcal{A}(Q) = \mathbf{T}$ and $\mathcal{A}(R) = \mathbf{F}$. Conjunctive normal form rules this possibility out:

$$\neg P \lor \neg Q \lor R$$

Tableau Proofs (15 points)

Prove the following tautologies using the semantic tableaux proofs.

$$\begin{array}{ll} (1) & (P \to Q) \to ((P \to (Q \to R)) \to (P \to R)) \\ (2) & ((P \to Q) \land (\neg P \to Q)) \to Q \\ (3) & ((P \to Q) \to P) \to P \\ (4) & ((P \to Q) \land (Q \to R)) \to \neg(\neg R \land P) \\ (5) & (P \leftrightarrow (P \to Q)) \leftrightarrow (P \land Q) \end{array}$$