Exercise 1

Which of the following expressions are official (that is, unabbreviated). Prove your answer by either giving a step by step derivation of the expression or proving, by induction on propositions, that there is some property enjoyed by all propositions but not this expression.

(a). (1pt)

$$\begin{aligned} prp \;&\Rightarrow\; (prp \wedge prp) \;\Rightarrow\; ((\neg\, prp) \wedge prp) \\ &\Rightarrow\; ((\neg\,(prp \vee prp)) \wedge prp) \;\Rightarrow\; ((\neg\,(P \vee Q)) \wedge R) \end{aligned}$$

(b). (3pt) This is not a proposition because it has more connectives then left parentheses, and all propositions have the same number of left parentheses as connectives.

Lemma. Every proposition has the same number of left parentheses as connectives.

Proof. Let S be the set of propositions with the same number of left parentheses as connectives.

BASIS. All propositional atoms have zero connectives and zero left parentheses. So, S contains all propositional atoms.

INDUCTIVE. Suppose $\alpha, \beta \in S$. Let α have k left parentheses and k connectives, and β have ℓ left parentheses and ℓ connectives. Then $(\neg\,\alpha)$ has $k+1$ left parentheses and $k+1$ connectives, so is in S. For each $\diamond \in \{\wedge\,,\,\vee\,,\,\rightarrow\,,\,\leftrightarrow\}$, $(\alpha \diamond \beta)$ has $k+\ell+1$ left parentheses and $k+\ell+1$ connectives, so is in S.

Thus, $S = \textbf{PROP}$. □

(c). (1pt)

$$\begin{aligned} prp \;&\Rightarrow\; (prp \leftrightarrow prp) \;\Rightarrow\; ((prp \wedge prp) \leftrightarrow prp) \\ &;\Rightarrow\; (((prp \vee prp) \wedge prp) \leftrightarrow prp);\Rightarrow\; (((Q \vee R) \wedge P) \leftrightarrow Q) \end{aligned}$$

Exercise 2

Define using structural recursion a function d on propositions, so that d returns the number of logical connectives occurring in the proposition. (If α is a propositional atom, then $d(\alpha) = 0$.)

(3 pts) Define $d : \textbf{PROP} \to \mathbb{N}$ by recursion:

$$\begin{aligned} &d(P) = 0 \qquad \text{for each propositional symbol } P \\ &d(\top) = d(\bot) = 1 \\ &d(\neg\,\alpha) = d(\alpha) + 1 \\ &d(\alpha \diamond \beta) = d(\alpha) + d(\beta) + 1 \qquad \text{for each } \diamond \in \{\,\wedge\,,\,\vee\,,\,\to\,,\,\leftrightarrow\,\}. \end{aligned}$$

We ought to prove that d really does count the number of connectives (although I did not ask you to do this). Let S be the set of propositions α such that $d(\alpha)$ is the number of connectives. If α is an atom, α has no logical connectives and $d(\alpha) = 0$ by the first two lines in the definition of d. So, S contains all atoms.

Suppose $\alpha, \beta \in S$. So, $d(\alpha)$ is the number of connectives in α and $d(\beta)$ is the number of connectives in β. Then $(\neg\,\alpha)$ has one more connective then α and $d(\neg\,\alpha) = d(\alpha) + 1$. So, $\neg\,\alpha \in S$. Let $\diamond \in \{\,\wedge\,,\,\vee\,,\,\to\,,\,\leftrightarrow\,\}$, so that $(\alpha \diamond \beta)$ has one more connective then the sum of the connectives in α and β. But $d(\alpha \diamond \beta) = d(\alpha) + d(\beta) + 1$, so $(\alpha \diamond \beta) \in S$. Thus, $S = \textbf{PROP}$.

Exercise 3

Which of the following are tautologies. It they are not a tautology, provide an assignment which establishes this. The truth table for $\uparrow$ (read as "not both" or NAND) is as follows:

P	Q	$P \uparrow Q$
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**

(a). (2pts) $((P \to Q) \wedge (Q \leftrightarrow \top)) \to (P \leftrightarrow \top)$ is not a tautology. Consider the assignment $\mathcal{A}(P) = \textbf{F}$ and $\mathcal{A}(Q) = \textbf{T}$. (This is the only possible assignment of the four on which the proposition is false.)

(b). (2pts) $(((\neg(P \wedge Q)) \wedge (P \leftrightarrow \bot)) \leftrightarrow (\neg(Q \leftrightarrow \bot)))$ is not a tautology. There are two possible assignments: $\mathcal{A}(P) = \mathcal{A}(Q) = \mathbf{T}$ and $\mathcal{A}(P) = \mathcal{A}(Q) = \mathbf{F}$.

(c). (2pts) $(((P \rightarrow Q) \uparrow (P \rightarrow Q)) \leftrightarrow (P \wedge \neg Q))$ is a tautology:

P	Q	$(((P \rightarrow Q) \uparrow (P \rightarrow Q)) \leftrightarrow (P \wedge \neg Q))$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**

(d). (2pts) $(((P \uparrow Q) \uparrow (P \uparrow Q)) \leftrightarrow (P \wedge Q))$ is a tautology:

P	Q	$(((P \uparrow Q) \uparrow (P \uparrow Q)) \leftrightarrow (P \wedge Q))$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**

(e). (2pt) There are lots of possibilities. The most straightforward is $\neg(P \wedge Q)$.

Exercise 4

The island of knights and knaves has the following laws: (i) everyone is a knight or knave, (ii) knights only speak truly, (iii) knaves only speak falsely.

Use the following translation key:

k_1: A_1 is a knight.

k_2: A_2 is a knight.

G: There is gold on the island.

and translate the following conversations as a pair of statements. In each case state which of the propositions k_1, k_2 and G must be true, which must be false, or either truth value is possible.

(a). (4pts)

A_1 If A_2 is a knight, then there is gold on the island.
$k_1 \leftrightarrow (k_2 \rightarrow G)$

A_2 If A_1 is a knave, then there is gold on the island.
$k_2 \leftrightarrow (\neg k_1 \rightarrow G)$

If A_1 is a knave, then A_2 would have to be a knight to and there is no gold on the island to make A_1's claim false. But there has to be be gold to make A_2's claim true. This is impossible. So, A_1 must be a knight. A_2 cannot be a knave, or A_2 would be speaking truly, since $\neg k_1 \rightarrow G$ has a false antecedent. So, A_2 must be a knight, and since A_1 is speaking truly, there must be gold on the island.

$\mathcal{A}(k_1) = \mathbf{T}, \mathcal{A}(k_2) = \mathbf{T}, \mathcal{A}(G) = \mathbf{T}$ is the only possible assignment.

(b). (4pts)

A_1 If I am a knight and A_2 is a knave, then there is gold on the island.
$k_1 \leftrightarrow ((k_1 \wedge \neg k_2) \rightarrow G)$

A_2 That is not true!
I accepted either $k_2 \leftrightarrow \neg((k_1 \wedge \neg k_2) \rightarrow G)$, or the simpler $k_2 \leftrightarrow \neg k_1$.

We know that A_1 and A_2 must be opposite types. If A_1 is a knave, then what A_1 says would be true, since $(k_1 \wedge \neg k_2)$ is false. So, A_1 must be a knight. Thus, A_2 is a knight, make $(k_1 \wedge \neg k_2)$ true. So, there is gold on the island.

$\mathcal{A}(k_1) = \mathbf{T}, \mathcal{A}(k_2) = \mathbf{F}, \mathcal{A}(G) = \mathbf{T}$ is the only possible assignment.

(c). (4pts)

A_1 If we are of the same type, then there is gold on the island.
$k_1 \leftrightarrow ((k_1 \leftrightarrow k_2) \rightarrow G)$.

A_2 We are not of the same type.
$k_2 \leftrightarrow \neg(k_1 \leftrightarrow k_2)$.

The quickest way to see this is to replace $(k_1 \leftrightarrow k_2)$ in A_1's statement with $\neg k_2$. So, the following must be true: $k_1 \leftrightarrow (\neg k_2 \rightarrow G)$. If A_1 was a knight, then A_2 could not be a knight (A_2 would be speaking falsely) and A_2 could not be a knave (A_2 would be speaking truly). So, A_1 must be a knave. Since A_1 speaks falsely, A_2 must be a knave and there is no gold.

$\mathcal{A}(k_1) = \mathbf{F}, \mathcal{A}(k_2) = \mathbf{F}, \mathcal{A}(G) = \mathbf{F}$ is the only possible assignment.