Math 425
Introduction to Probability
Lecture 8

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Conditional Probability

☞ Very few experiments amount to just one action with random outcomes.

- Sometimes conditions change before the experiment is completed.
- Some experiments have a more complicated structure, in which some partial results are known before the experiment is completed.

☞ We need a rule for recomputing probabilities when the conditions of an experiment change.
This new rule is call the Conditioning Rule.

Example 1

Example

☞ You are about to roll a red and blue die.
What is the probability that the red die is larger?

$$\mathbf{P}\,(\textit{red larger}) = \frac{15}{36}.$$

☞ The blue die is first rolled and shows 5.
What is the probability that the red die is larger?

$$\mathbf{P}\,(\textit{red larger}) = \frac{1}{6}.$$

Example 2: Kidney stones

Example

Kidney stones are either small ($<$ 2 cm diameter) or large ($>$ 2 cm diameter).
Here are the outcomes of one treatment:

Size	**Outcomes**		
	success (C)	failure	**total**
small	315	42	357
large (L)	247	96	343
total	562	138	700

Example 2: Kidney stones

☞ For a patient picked at random from the 700 patients

$$\mathbf{P}(L) = \frac{343}{700}.$$

☞ For a patient picked at random, the probability of success is

$$\mathbf{P}(C) = \frac{562}{700} \approx 0.8.$$

☞ For a patient picked at random, among those with large stones, the probability of success is

$$\frac{247}{343} \approx 0.72.$$

Notation

☞ We introduce a notation to make explicit the new conditions:

$$\mathbf{P}(C \mid L) \approx 0.72$$

means the probability of C (success) given L (large stones).

☞ Suppose we select a patient at random, among those with successful treatments, to determine if their stones were large.

$$\mathbf{P}(L \mid C) = \frac{247}{562} \approx 0.44.$$

$\mathbf{P}(L \mid C)$ is the conditional probability of L (large stones) given C (success).

Conditional probability

Definition

Let E and F be events.
The conditional probability of E given F is defined by:

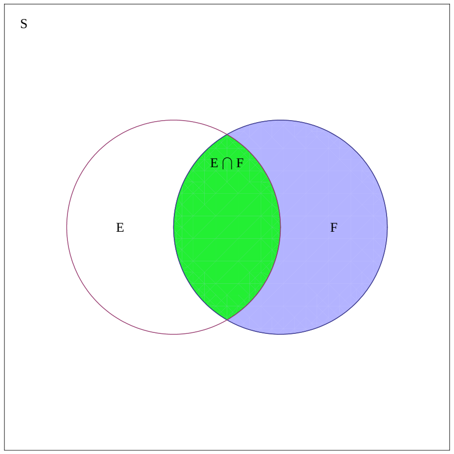
$$\mathbf{P}(E \mid F) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)}.$$

provided $\mathbf{P}(F) > 0$.

Conditional probability – picture

Conditional probability shrinks the sample space S to an event F:

$$\mathbf{P}(E \mid F) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)}$$



Example 1 – revisited

☞ Let R be the event that the red die is bigger, and B be the event that the blue die shows 5.

According to the rule,

$$\mathbf{P}(R \mid B) = \frac{\mathbf{P}(R \cap B) = \frac{1}{36}}{\mathbf{P}(B) = \frac{1}{6}} = \frac{1}{6}$$

Note that $R \cap B$ has one outcome: $(5, 6)$.

Example 2 – revisited

Size	**Outcomes**		
	success (C)	failure	**total**
small	315	42	357
large (L)	247	96	343
total	562	138	700

☞ According to the rule,

$$\mathbf{P}(C \mid L) = \frac{\mathbf{P}(C \cap L) = \frac{247}{700}}{\mathbf{P}(L) = \frac{343}{700}} = \frac{247}{343}$$

Example 3

Example

A fair coin is flipped three times. Consider the events

- A: the first flip is heads.
- B: there are exactly two heads overall.

Then,

$$\begin{aligned} S &= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \\ A &= \{HHH, HHT, HTH, HTT\} \\ B &= \{HHT, HTH, THH\} \\ A \cap B &= \{HHT, HTH\}. \end{aligned}$$

Apply the conditioning rule to compute,

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{2}{3} \qquad \mathbf{P}(B \mid A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{2}{4}.$$

Example 4

Example

A box contains a double-headed coin, a double-tailed coin and a conventional coin. One coin is picked at random, and shows heads.

What is the probability that it is the double-headed coin?

☠ Warning. The following reasoning is incorrect:

- Since the coin shows heads, it is either double-headed or conventional. Since the coin was picked at random, these are equally likely, so the probability is $\frac{1}{2}$.

This is wrong!!

Example 4 – continued

☞ Let the sample space be the 6 possible faces on the three coins. Consider the events

- H: coin shows heads,
- D: coin is double-headed.

Since 3 faces yield H,

$$\mathbf{P}(H) = \frac{1}{2} \qquad \mathbf{P}(H \cap D) = \frac{1}{3}.$$

Apply the conditioning rule,

$$\mathbf{P}(D \mid H) = \frac{\mathbf{P}(H \cap D)}{\mathbf{P}(H)} = \frac{2}{3}.$$

Example 4 – Sample space

☞ Consider the sample space, where the outcomes are H, T and I (double-tailed), II (double-headed) and III (Normal):

$$S = \{(I, T_1), (I, T_2), (II, H_1), (II, H_2), (III, H_1), (III, T_2)\}$$

All outcomes are equiprobable.

☞ After we see a heads, the sample space is reduced:

$$\{(I, T_1), (I, T_2), (II, H_1), (II, H_2), (III, H_1), (III, T_2)\}$$

The outcomes remaining are still equiprobable.

$$\mathbf{P}(D \mid H) = \frac{2}{3}.$$

Example 4 – Simulation

☞ I ran a simulation of Example 4 in Mathematica.

1. Choose a coin at random and simulate a toss.
2. If Tails shows, go on to next trial.
3. If Heads shows, record which coin was chosen.
4. After 10,000 trials, output

$$\frac{|D \cap H|}{|H|}$$

☞ I ran the simulation 1000 times (our calculation was 0.666667):

- Mean: 0.667024 (of ratios)
- Maximum ratio: 0.688831
- Minimum ratio: 0.645387
- Standard Deviation: 0.006673

You can be 95% confident that the correct probability is within two decimal places of the mean.

Conjunctions

☞ We can often compute $\mathbf{P}(F)$ and $\mathbf{P}(E \mid F)$ easily. If so, we can compute the conjunction $\mathbf{P}(E \cap F)$ as well.

Lemma (Multiplication Rule)

For any events E and F (where $\mathbf{P}(F) > 0$),

$$\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F \mid E)$$

Example 5

Example

A drawer contains 5 red socks and 3 blue socks.
If you remove the socks at random, what is the probability of holding a blue pair?

☞ You could do this by enumerating all the possible combinations in the sample space for this experiment, but conditioning provides a simpler method.

Example 5 – continued

☞ 5 red and 3 blue socks.
Consider the two events

- B_i: the ith sock picked is blue ($i = 1, 2$).

We want to determine $\mathbf{P}(B_1 \cap B_2)$.
Use the multiplication rule,

$$\begin{aligned}\mathbf{P}(B_1 \cap B_2) &= \mathbf{P}(B_1) \cdot \mathbf{P}(B_2 \mid B_1) \\ &= \frac{3}{8} \cdot \frac{2}{7} = \frac{3}{28}.\end{aligned}$$

Reason: If you choose 1 blue sock then there are 2 blue socks and 7 socks remaining. So,

$$\mathbf{P}(B_2 \mid B_1) = \frac{2}{7}.$$

Multiplication Rule for 3 events

☞ We can extend the Multiplication Rule to three events.

Lemma

For any events E, F, G (provided $\mathbf{P}(E \cap F \cap G) > 0$)

$$\mathbf{P}(E \cap F \cap G) = \mathbf{P}(E) \cdot \mathbf{P}(F \mid E) \cdot \mathbf{P}(G \mid E \cap F)$$

Proof. Use the Mutiplication Rule twice,

$$\begin{aligned}\mathbf{P}(E \cap F \cap G) &= \mathbf{P}(E \cap F) \cdot \mathbf{P}(G \mid E \cap F) \\ &= \mathbf{P}(E) \cdot \mathbf{P}(F \mid E) \cdot \mathbf{P}(G \mid E \cap F)\end{aligned}$$

We need $\mathbf{P}(E \cap F \cap G) \neq 0$ to ensure the conditional probabilities exist.

Example 6

Example

An urn is filled with 6 red balls, 5 blue balls, and 4 green balls. Three balls chosen at random are removed from the urn.
What is the probability that the balls are of the same color?

☞ We are interested in the events (where $i = 1, 2, 3$)

- R_i: ith ball drawn is red,
- B_i: ith ball drawn is blue,
- G_i: ith ball drawn is green.
- C: three balls are the same color.

Example 6 – continued

☞ Urn: 6 red, 5 blue, and 4 green. Use the Multiplication Rule,

$$\begin{aligned}\mathbf{P}(R_1 \cap R_2 \cap R_3) &= \mathbf{P}(R_1) \cdot \mathbf{P}(R_2 \mid R_1) \cdot \mathbf{P}(R_3 \mid R_1 \cap R_2) \\ &= \frac{6}{15} \cdot \frac{5}{14} \cdot \frac{4}{13} = \frac{4}{91}\end{aligned}$$

$$\begin{aligned}\mathbf{P}(B_1 \cap B_2 \cap B_3) &= \mathbf{P}(B_1) \cdot \mathbf{P}(B_2 \mid B_1) \cdot \mathbf{P}(B_3 \mid B_1 \cap B_2) \\ &= \frac{5}{15} \cdot \frac{4}{14} \cdot \frac{3}{13} = \frac{2}{91}\end{aligned}$$

$$\begin{aligned}\mathbf{P}(G_1 \cap G_2 \cap G_3) &= \mathbf{P}(G_1) \cdot \mathbf{P}(G_2 \mid G_1) \cdot \mathbf{P}(G_3 \mid G_1 \cap G_2) \\ &= \frac{4}{15} \cdot \frac{3}{14} \cdot \frac{2}{13} = \frac{4}{455}\end{aligned}$$

Since these events are mutually exclusive,

$$\mathbf{P}(C) = \frac{4}{91} + \frac{2}{91} + \frac{4}{455} = \frac{34}{455} \approx 0.0747.$$

General multiplication Rule

☞ The Multiplication rule is the probabilistic version of the product rule for counting.

Theorem (Generalized Multiplication Rule)

Let $E_1, E_2, \ldots, E_n$ be any events such that

$$\mathbf{P}(E_1 \cap E_2 \cap \cdots \cap E_n) > 0.$$

Then

$$\begin{aligned}\mathbf{P}(E_1 \cap E_2 \cap \ldots \cap E_n) &= \mathbf{P}(E_1) \cdot \mathbf{P}(E_2 \mid E_1) \cdot \mathbf{P}(E_3 \mid E_1 \cap E_2) \cdots \\ &\quad \cdots \mathbf{P}(E_n \mid E_1 \cap E_2 \cap \ldots \cap E_{n-1}).\end{aligned}$$

(See Ross p. 71.)

Example 7

Example

In Pick-Six Lottery: A person purchases a ticket, and can choose 6 distinct numbers in the set $\{1, 2, 3, \ldots, 49\}$.

Later a Lottery Machine picks 6 distinct numbers at random in the set $\{1, 2, 3, \ldots, 49\}$.

A winning ticket is one which matches the six numbers chosen by the Machine (in any order of selection).

☞ What are the odds of winning with one ticket?

Example 7 – solution

Solution. We solved this before (Lecture 5) by counting using the Product Rule for counting. We conditionalize here.

☞ Let E_i be the event that there are i matches. We want to compute

$$\begin{aligned}\mathbf{P}(E_6) &= \mathbf{P}(E_1 \cap E_2 \cap \ldots \cap E_6) \\ &= \mathbf{P}(E_1) \cdot \mathbf{P}(E_2 \mid E_1) \cdots \mathbf{P}(E_6 \mid E_1 \cap \ldots \cap E_5) \\ &= \frac{6}{49} \cdot \frac{5}{48} \cdot \frac{4}{47} \cdot \frac{3}{46} \cdot \frac{2}{45} \cdot \frac{1}{44} \\ &= \frac{1}{13,983,816}\end{aligned}$$