Math 425
Introduction to Probability
Lecture 7

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Unions and Intersections

✓ This lecture will focus on computing the probability of arbitrary finite unions of events

$$\mathbf{P}(E_1 \cup E_2 \cup \ldots \cup E_n)$$

even when the events are not mutually inclusive.

The method is called the Inclusion-Exclusion Principle (Proposition 2.4.4).

✓ Chapter 3 (the next five lectures) will introduce methods for computing the probability of arbitrary finite intersections of events

$$\mathbf{P}(E_1 \cap E_2 \cap \ldots \cap E_n).$$

The method is called the Conditionalization.

Proposition 4.3

☞ The Inclusion-Exclusion Principle (for $n = 2$) is used to compute

$$\mathbf{P}(E \cup F)$$

when E and F may not be mutually exclusive.

Proposition (Ross 4.3)

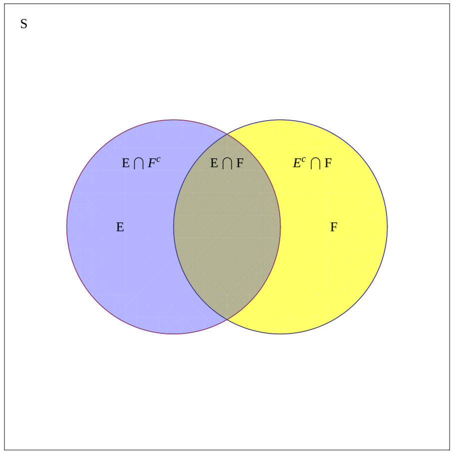
Let E and F be any events. Then

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$$

Picture Proof of Proposition 4.3

Picture Proof.

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$$



Example 0

Example 0. What is the probability of getting a heads on the first toss or a tails on the second toss of a fair coin?

Sample Space.

$$S = \{HH, HT, TH, TT\}$$

Assumption. Each outcome is equiprobable.

Events.

- H_i: heads on ith toss ($i = 1, 2$),
- T_i: tails on ith toss ($i = 1, 2$)

Probabilities. We want to compute $\mathbf{P}(H_1 \cup T_2)$.

$$\mathbf{P}(H_i) = \frac{1}{2} \qquad \mathbf{P}(T_i) = \frac{1}{2} \qquad \mathbf{P}(H_1 \cap T_2) = \frac{1}{4}$$

$$\mathbf{P}(H_1) + \mathbf{P}(T_2) = 1 \qquad \mathbf{P}(H_1 \cup T_2) = \frac{3}{4}.$$

Proposition 4.4, Case $n = 3$

☞ The Inclusion-Exclusion Principle (for $n = 3$) is used to compute

$$\mathbf{P}(E \cup F \cup G)$$

when E, F and G may not be mutually exclusive.

Proposition (Ross 4.4, case $n = 3$)

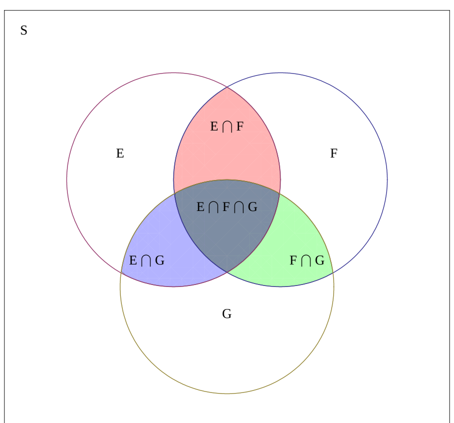
Let E, F and G be any events. Then

$$\begin{aligned} \mathbf{P}(E \cup F \cup G) &= \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ &\quad - (\mathbf{P}(E \cap F) + \mathbf{P}(E \cap G) + \mathbf{P}(F \cap G)) \\ &\quad + \mathbf{P}(E \cap F \cap G) \end{aligned}$$

Picture Proof of case $n = 3$

Picture Proof

$$\begin{aligned} \mathbf{P}(E \cup F \cup G) &= \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ &\quad - (\mathbf{P}(E \cap F) + \mathbf{P}(E \cap G) + \mathbf{P}(F \cap G)) \\ &\quad + \mathbf{P}(E \cap F \cap G) \end{aligned}$$



Example 0

Example 0. What is the probability of getting a heads on the first toss or a tails on the second toss or heads on the third toss of a fair coin?

Sample Space.

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Assumption. Each outcome is equiprobable.

Events.

- H_i: heads on ith toss ($i = 1, 2, 3$),
- T_i: tails on ith toss ($i = 1, 2, 3$)

Probabilities. We want to compute $\mathbf{P}(H_1 \cup T_2 \cup H_3)$.

$$\mathbf{P}(H_i) = \frac{1}{2} \qquad \mathbf{P}(T_i) = \frac{1}{2}$$

$$\mathbf{P}(H_1 \cap T_2 \cap H_3) = \frac{1}{8} \qquad \mathbf{P}(H_i \cap T_j) = \frac{1}{4} \text{ where } i \neq j$$

$$\mathbf{P}(H_1) + \mathbf{P}(T_2) + \mathbf{P}(H_3) = \frac{3}{2} \qquad \mathbf{P}(H_1 \cup T_2 \cup H_3) = \frac{7}{8}.$$

Example 1

Example 1. What is the probability that a randomly chosen integer between 1 and 10,000 is divisible by at least one of 4, 10 or 25?

☞ The sample space is integers between 1 and 10,000.
We interested in the events

- E_k: the chosen number is divisible by k (where $k = 4, 10, 25$.)

Assumption. Every outcome is equiprobable:
the probability of a number being chosen is $\frac{1}{10000}$.

Example 1 – continued

☞ We want to compute $\mathbf{P}(E_4 \cup E_{10} \cup E_{25})$, and since these events are NOT mutually exclusive (why?), we need to use the Inclusion-Exclusion Principle.

☞ The following are easy to compute:

$$\mathbf{P}(E_4) = \frac{1}{4} \quad \mathbf{P}(E_{10}) = \frac{1}{10} \quad \mathbf{P}(E_{25}) = \frac{1}{25}.$$

☞ In general, if k evenly divides 10,000:

$$|E_k| = \frac{10000}{k}, \qquad \text{so } \mathbf{P}(E_k) = \frac{1}{k}.$$

Example 1 – continued

☞ We also need to compute the probabilties of the intersections.

$$\begin{aligned}
E_4 \cap E_{10} &= E_{20} && \text{since } 20 = \mathrm{lcm}(4, 10) \\
E_4 \cap E_{25} &= E_{100} && \text{since } 100 = \mathrm{lcm}(4, 25) \\
E_{10} \cap E_{25} &= E_{50} && \text{since } 50 = \mathrm{lcm}(4, 10) \\
E_4 \cap E_{10} \cap E_{25} &= E_{100} && \text{since } 100 = \mathrm{lcm}(4, 10, 25)
\end{aligned}$$

☞ By the Inclusion-Exclusion Principle:

$$\begin{aligned}
\mathbf{P}(E_4 \cup E_{10} \cup E_{25}) &= \frac{1}{4} + \frac{1}{10} + \frac{1}{25} - \frac{1}{20} - \frac{1}{50} - \frac{1}{100} + \frac{1}{100} \\
&= \frac{32}{100}.
\end{aligned}$$

Example 2

Example 2. From a faculty of 5 full professors, 7 associate professors and 11 assistant professors, a committee of size 4 is to be formed randomly.

What is the probability that a person from each rank is represented on the committee?

☠ Warning. The following is not correct (it overcounts groups):

$$\frac{\binom{5}{1} \cdot \binom{7}{1} \cdot \binom{11}{1} \cdot \binom{20}{1}}{\binom{23}{4}} = \frac{20}{23} \approx 0.870!!$$

More than half the committees are formed from only two of the three groups:

$$\frac{\binom{12}{4} + \binom{16}{4} + \binom{18}{4}}{\binom{23}{4}} = \frac{1075}{1771} \approx 0.6070$$

Example 2 – continued

- Apply the Inclusion-Exclusion Principle to avoid overcounting.
- Let the sample space S consist of sets

$$\{p_1, p_2, p_2, p_4\} \quad \text{where } 1 \le p_1, p_2, p_3, p_4 \le 23,$$

and these numbers mean:

$$1 \le p \le 5: \text{ [full]} \quad 6 \le p \le 12: \text{ [associate]} \quad 13 \le p \le 23: \text{ [assistant]}.$$

- We are interested in the following events:
  - E_1: NO full professor is on the committee,
  - E_2: NO associate professor is on the committee,
  - E_3: NO assistant professor is on the committee,
  - F: All three positions are represented on the committee.

Example 2 – continued

- We want to compute $\mathbf{P}(F)$. We can do this by as follows.
  The event

$$E_1 \cup E_2 \cup E_3$$

is that of all committees that are missing some group. So,

$$F = (E_1 \cup E_2 \cup E_3)^c,$$

and

$$\mathbf{P}(F) = 1 - \mathbf{P}(E_1 \cup E_2 \cup E_3).$$

- The events E_1, E_2, E_3 are NOT mutually exclusive, so we need to use the Inclusion-Exclusion Principle to compute

$$\mathbf{P}(E_1 \cup E_2 \cup E_3).$$

Example 2 – continued

- Apply the Inclusion-Exclusion Principle.

$$\begin{aligned}
|S| &= \binom{23}{4} & |E_1 \cap E_2| &= \binom{11}{4} \\
|E_1| &= \binom{18}{4} & |E_1 \cap E_3| &= \binom{7}{4} \\
|E_2| &= \binom{16}{4} & |E_2 \cap E_3| &= \binom{5}{4} \\
|E_3| &= \binom{12}{4} & |E_1 \cap E_2 \cap E_3| &= 0
\end{aligned}$$

So,

$$\begin{aligned}
\mathbf{P}(F) &= 1 - \frac{\binom{18}{4} + \binom{16}{4} + \binom{12}{4} - \binom{11}{4} - \binom{7}{4} - \binom{5}{4}}{\binom{23}{4}} \\
&= \frac{10}{23} \approx 0.435.
\end{aligned}$$

Generalized Inclusion-Exclusion Principle

- The Inclusion-Exclusion Principle for 3 events E_1, E_2, E_3:

$$\mathbf{P}(E_1 \cup E_2 \cup E_3) = \Sigma_1 - \Sigma_2 + \Sigma_3,$$

where

$$\begin{aligned}
\Sigma_1 &= \mathbf{P}(E_1) + \mathbf{P}(E_2) + \mathbf{P}(E_3) \\
\Sigma_2 &= \mathbf{P}(E_1 \cap E_2) + \mathbf{P}(E_1 \cap E_3) + \mathbf{P}(E_2 \cap E_3) \\
\Sigma_3 &= \mathbf{P}(E_1 \cap E_2 \cap E_3).
\end{aligned}$$

Σ_i ($i = 1, 2, 3$) is the sum of all intersections taken i at a time.

Generalized Inclusion-Exclusion Principle

☞ The general Inclusion-Exclusion Principle is used for computing

$$\mathbf{P}(E_1 \cup E_2 \cup \ldots \cup E_n)$$

for any events $E_1, E_2, \ldots, E_n$ (even if they are not mutually exclusive).

Theorem (Ross Proposition 4.4 – Inclusion-Exclusion Principle)

Let $E_1, E_2, \ldots, E_n$ be any events. Then

$$\mathbf{P}(E_1 \cup E_2 \cup \ldots \cup E_n) = \Sigma_1 - \Sigma_2 + \Sigma_3 - \ldots + (-1)^{n+1}\Sigma_n$$

where Σ_r is the sum of the $\binom{n}{r}$ terms

$$\mathbf{P}(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_r})$$

over all subsets $\{i_1, i_2, \ldots, i_r\}$ of size r from $\{1, 2, \ldots, n\}$.

Example 1

(See Ross, Exercise 2.5n.)

Example 1. Eight people (forming 4 couples) are sat at random at a round table.

What is the probability that no couple sits together?

☞ There are 7! ways of seating 8 people at a round table (where there is no privileged position, such as head of the table).

Assumption. All possible arrangements are equiprobable.

Example 1 – continued

☞ We are interested in the events

- E_i: the ith couple sits together (where $i = 1, 2, 3, 4$).
- F: NO couple sits together.

Thus,

$$F = (E_1 \cup E_2 \cup E_3 \cup E_4)^c.$$

and we want to compute the probability

$$\mathbf{P}(F) = 1 - \mathbf{P}(E_1 \cup E_2 \cup E_3 \cup E_4).$$

☞ The events E_1, E_2, E_3, E_4 are NOT mutually exclusive.

Example 1 – continued

☞ Compute the probability of each E_i ($i = 1, 2, 3, 4$):

$$\mathbf{P}(E_i) = \frac{|E_i|}{|S|} = \frac{2 \cdot 6!}{7!}.$$

Reason:

- Treat couple i as one person, so that we need to seat 7 people at a round table: 6! choices.
- For each arrangement, we have two ways to seat the couple (expanding their one spot to two).

Example 1 – continued

☞ Compute each of the intersections using the same argument (condensing a couple to one person):

$$\begin{aligned}
\mathbf{P}(E_i) &= \frac{2\cdot 6!}{7!} \quad i=1,2,3,4\\
\mathbf{P}(E_i\cap E_j) &= \frac{2^2\cdot 5!}{7!} \quad 1\le i<j\le 4\\
\mathbf{P}(E_i\cap E_j\cap E_k) &= \frac{2^3\cdot 4!}{7!} \quad 1\le i<j<k\le 4\\
\mathbf{P}(E_1\cap E_2\cap E_3\cap E_4) &= \frac{2^4\cdot 3!}{7!}.
\end{aligned}$$

These values do not depend on i, j, k.

Example 1 – completed

☞ Use the Inclusion-Exclusion Principle to complete the computation:

$$\begin{aligned}
\mathbf{P}(F) &= 1-\Sigma_1+\Sigma_2-\Sigma_3+\Sigma_4\\
&= 1-\binom{4}{1}\cdot\frac{2\cdot 6!}{7!}+\binom{4}{2}\cdot\frac{2^2\cdot 5!}{7!}-\binom{4}{3}\frac{2^3\cdot 4!}{7!}+\binom{4}{4}\frac{2^4\cdot 3!}{7!}\\
&= \frac{31}{105}\approx 0.2952.
\end{aligned}$$

Example 2

(See Ross, Exercise 2.5m.)

Example 2. A group of n musicians from an orchestra are exchanging names for gift-giving at the end of the season bash. Each places their name in an urn, which is completely mixed, then draw names one-by-one.

Question. What is the probability that no one draws their own name?

Example 2 – Sample space

☞ The sample space S is the set of all permutations on $\{1, 2, \ldots, n\}$, where we replaced the name of each musician with a number. So, a permutation

$$(i_1, i_2, \ldots, i_n)$$

represents musician k draws i_k.

There are $n!$ possible outcomes of the drawing.

Assumption. All permutations are equiprobable, since the names were well mixed.

Example 2 – Events

☞ We are interested in the following events.

- E_k: the kth musician draws their own name. These outcomes are

$$(i_1, i_2, \ldots, i_n) \qquad \text{where } i_k = k.$$

- F: NO musician draws their name.

☞ The event F is

$$F = (E_1 \cup E_2 \cup \ldots \cup E_n)^c;$$

so, the probability we want is

$$\mathbf{P}(F) = 1 - \mathbf{P}(E_1 \cup E_2 \cup \ldots \cup E_n).$$

The events $E_1, E_2, \ldots, E_n$ are NOT mutually exclusive.

Example 2 – Events

☞ Compute $\mathbf{P}(E_1 \cup E_2 \cup \ldots \cup E_n)$ using the Inclusion-Exclusion Principle.

☞ For any k,

$$\mathbf{P}(E_k) = \frac{(n-1)!}{n!},$$

☞ Suppose $k \neq \ell$, then

$$\mathbf{P}(E_k \cap E_\ell) = \frac{(n-2)!}{n!},$$

☞ More generally, for any subset $\{i_1, i_2, \ldots, i_r\}$ (where $r \leq n$):

$$\mathbf{P}(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_r}) = \frac{(n-r)!}{n!}.$$

This is same value for any subset of size r.

Example 2 – Events

☞ For each $r \leq n$, there are $\binom{n}{r}$ subsets of size r, and each has the same probability. So,

$$\Sigma_r = \frac{n!}{(n-r)! \cdot r!} \cdot \frac{(n-r)!}{n!} = \frac{1}{r!}.$$

☞ By the Inclusion-Exclusion Principle

$$\begin{aligned} \mathbf{P}(F) &= 1 - \mathbf{P}(E_1 \cup E_2 \cup \ldots \cup E_n) \\ &= 1 - \Sigma_1 + \Sigma_2 - \ldots + (-1)^n \Sigma_n \\ &= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} \end{aligned}$$

Example 2 – concluded

☞ For n musicians, the probability that NO two pick their own name is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}.$$

☞ Compare this to the power series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots,$$

and you get as $n \to \infty$,

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} \to e^{-1} \approx 0.36788!!$$

☞ So, the probability that no musicians draw their name converges to e^{-1} as $n \to \infty$.

Example 3

Example 3. What is the probability that exactly k musicians from Example 2 draw their own name?

☞ The probability that musicians $i_1, i_2, \ldots, i_k$ draw their own name is

$$\mathbf{P}\left(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}\right) = \frac{(n-k)!}{n!}.$$

independently of the choice of musicians $i_1, i_2, \ldots, i_k$.

However, this leaves open the possibility that other musicians draw their own name.

Example 3 – continued

☞ P_r be the probability that none of r musicians draw their own name:

$$P_r = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!}.$$

So, the number of ways these musicians can draw is $r! \cdot P_r$.

☞ The number of ways that exactly k musicians $i_1, i_2, \ldots, i_k$ selected from n draw their own name is (none of the $n-k$ others draw theirs)

$$(n-k)! \cdot P_{n-k}$$

The probability that only these k musicians draw their own name is

$$\frac{(n-k)! \cdot P_{n-k}}{n!}.$$

This is the same probability for any k musicians selected.

Example 3 – continued

☞ The probability that exactly k musicians are selected is

$$\begin{aligned}\binom{n}{k}\frac{(n-k)! \cdot P_{n-k}}{n!} &= \frac{n!}{(n-k)!k!} \cdot \frac{(n-k)! \cdot P_{n-k}}{n!} \\ &= \frac{P_{n-k}}{k!} \\ &= \frac{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{(n-k)}\frac{1}{(n-k)!}}{k!}\end{aligned}$$

☞ So, the probability that exactly k musicians draw their name converges to $\frac{e^{-1}}{k!}$ as $n \to \infty$.