Math 425
Introduction to Probability
Lecture 6

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The probability model

☞ The basis of probability is the experiment:

- An experiment is a repeatable procedure that has a a measurable outcome which cannot be predicted ahead of time.

☞ The probability model consists of three ingredients

1. A sample space S of possible outcomes of an experiment,
2. A collection $\mathcal{E}$ of events, which are subsets of S, and are said to occur as the outcome of an experiment.
3. A probability measure **P** which assigns a nonnegative real number to events (the "probability of the event occurring as the outcome of an experiment").

Enter Tyche

☞ A probability measure on a sample space S is a function **P** defined on the events E of the space S. We write

$\mathbf{P}(E)$ to mean the probability of the event E occuring

Sample space. Whenever we talk about a probability measure **P**, we are assuming it is a measure with respect to some given sample space S.

☞ A probability measure must also satisfy the three axioms which follow.

Axioms

Axiom (1 – Nonnegative)

For any event E, $\mathbf{P}(E)$ is a real number and

$$0 \leq \mathbf{P}(E).$$

Axiom (2 – Unit measure)

$$\mathbf{P}(S) = 1.$$

Axiom (3 – Addition rule (weak form))

For any mutually exclusive events E and F (so, $E \cap F = \emptyset$),

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F).$$

Axiom 3 – Strong form

Axiom (3 – Addition rule (strong form))

For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (so, $E_i \cap E_j = \emptyset$ whenever $i \neq j$),

$$\boldsymbol{P}\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \boldsymbol{P}(E_k).$$

☞ This rule is only relevant when the sample space S is infinite. The Strong form of the Addition Rule really is stronger than its Weak form. (See Ross, p. 30).

Finite sample spaces, equiprobable outcomes

☞ Let S is a finite sample space with N outcomes:

$$S = \{a_1, a_2, \ldots, a_N\}$$

☞ Suppose every outcome is equiprobable:

$$\mathbf{P}(\{a_1\}) = \mathbf{P}(\{a_2\}) = \ldots = \mathbf{P}(\{a_N\})$$

☞ Then it follows from the Axioms that:

1. For every $i \leq N$,
$$\mathbf{P}(\{a_i\}) = \frac{1}{N},$$
2. For any event E,
$$\mathbf{P}(E) = \frac{|E|}{N},$$
where $|E|$ is the number of outcomes in E.

Probability distribution

Definition

Let S be a discrete sample space with N outcomes

$$S = \{a_1, a_2, \ldots, a_N\}.$$

A probability distribution (or probability mass) on S is a function m on S satisfying:

(a) $m(a_i) \geq 0$ for all $i \leq N$,

(b) $m(a_1) + p(a_2) + \ldots + m(a_N) = 1$.

Note. We will come back to this concept in Chapter 4.

Examples of probability distributions

Examples

- The probability distribution m for a fair coin is one with $m(H) = m(T) = \frac{1}{2}$.
- The probability distribution m for a biased coin is one with $m(H) = p$ and $m(T) = 1 - p$ where $p \neq \frac{1}{2}$.
- The probability distribution m for a fair die is one which assigns every outcome the same probability: $m(i) = \frac{1}{6}$ where $1 \leq i \leq 6$.

Probability distribution

Theorem

Let S be a finite sample space and m a probability distribution. Define a function $\boldsymbol{P}$ *on the events E of S as follows:*

$$\boldsymbol{P}(E) = \sum_{a \in E} m(a)$$

Then $\boldsymbol{P}$ *is a probability function. (That is,* $\boldsymbol{P}$ *satisfies the probability axioms).*

Sixes wild

☞ Gambling houses in seventeenth century France offered the following game:

- The house (the management of the gambling establishment) offers to bet even money that a player will throw at least one six in 4 throws of a single six-sided die.

☞ A variation of the game was as follows

- The house offers to bet even money that a player will throw at least one double six in 24 throws of a pair of dice.

Old Gambler's Rule

☞ There was an old gambling rule which argued that if the first bet was favorable to the house (the house makes money) then the second bet should be favorable to the house:

1. At least one six in 4 throws of a single die,
2. At least one double six on 24 throws of a pair of dice.

Here is the argument:

- Throwing a double six on two die is six times less likely than a single six in one die. **True**
- To compensate, the two die should be thrown 6 times.
- And to match the probability of at least one six in 4 throws, the number of throws should be increased four fold – to 24.

The Chevalier de Méré

☞ Antoine Gombaud, Chevalier de Méré was a French writer and amateur mathematician of the seventeenth century, who also happened to be a professional gambler.

☞ de Méré suspected that the second game (played with two die) was unfavorable to the house, but that allowing 25 throws (instead of 24) turned the odds back to the houses favor.

☞ de Méré's experience seemed to contradict a valid observation (the old gambler's rule).

Enter Blaise Pascal

☞ de Méré turned to the pre-eminent mathematicians of his day, with the challenge to resolve this problem.

☞ Blaise Pascal and Pierre de Fermat took up the challenge in a series of letter, laying the mathematical foundations for the modern theory of probability.

Question. Was de Méré's suspicions correct?
What is the probability of each event:

1. At least one six in 4 throws of a single die,
2. At least one double six on 24 throws of a pair of dice.

Sample space: the modern approach

☞ We need to sample spaces for each of the two experiments.

- Let S_1 be the set of possible outcomes consisting of sequences

$$(a_1, a_2, a_3, a_4) \quad \text{where each } a_i = 1, 2, 3, 4, 5, 6.$$

  Note that $|S_1| = 6^4$ (the number of outcomes in S_1).

- Let S_2 be the set of possible outcomes consisting of sequences

$$((b_1, c_1), (b_2, c_2), \ldots, (b_{24}, c_{24})) \quad \text{where each } b_i, c_i = 1, 2, 3, 4, 5, 6.$$

  Note that $|S_2| = 36^{24}$ (the number of outcomes in S_2).

Assumption. The game is fair: Every outcome is equiprobable.

First problem

First problem. What is the probability of

- E: At least one six in 4 throws of a single die?

☞ We want to compute

$$\mathbf{P}(E) = 1 - \mathbf{P}(E^c).$$

E^c is that event that NO sixes are thrown. So,

$$\mathbf{P}(E^c) = \frac{|E^c|}{|S_1|} = \frac{5^4}{6^4},$$

since no six is thrown on any of the 4 throws.

☞

$$\mathbf{P}(E) = 1 - (\frac{5}{6})^4 = 0.517747.$$

Second Problem

Second problem. What is the probability of the event:

- F: At least one double six on 24 throws of a pair of dice?

☞ We want to compute

$$\mathbf{P}(F) = 1 - \mathbf{P}(F^c).$$

F^c is that event that NO sixes are thrown. So,

$$\mathbf{P}(F^c) = \frac{|F^c|}{|S_2|} = \frac{35^{24}}{36^{24}},$$

since no double sixes are thrown on any of the 24 throws.

☞

$$\mathbf{P}(F) = 1 - (\frac{35}{36})^{24} = 0.491404.$$

Problem resolved

Our computed probabilities

1. At least one six in 4 throws of a single die: 0.517747 ,
2. At least one double six on 24 throws of a pair of dice: 0.491404 .

☞ If the house allowed 25 throws instead of 24 throws, the probability would be back in the house's favor:

$$1 - \left(\frac{35}{36}\right)^{25} = 0.505532,$$

This is just what Pascal reported to de Méré.

Birthday Problem

Problem. What is the fewest number of people in a group before the probability exceeds $\frac{1}{2}$ that two or more of them have the same birthday?

(Compare to example (2.5i) in Ross.)

Note. We will ignore February 29 as a possible date, and treat the other 365 days as equally likely.

Reason: February 29 is not as likely to be a birthday. Is there any reason for thinking that some days are more likely to be birthdays than others?

Birthday Problem – continued

☞ We generalize the problem: Given r people in a group, what is the probability that two share the same birthday.

We assume $r \leq 365$. Why?

☞ Number each person 1 to r. Let the sample space S consist of all possible sequences of birthdates for r people:

$$(d_1, d_2, \ldots, d_r) \quad \text{where } d_1, d_2, \ldots, d_r \leq 365.$$

d_i is the birthdate of the ith person.

☞ There are 365^r outcomes in S.

Assumption. The distribution of birth dates is uniform, so each outcome is equiprobable.

Birthday Problem – continued

☞ Let E be the event that at least two dates are the same. The outcomes in E are sequences:

$$(d_1, d_2, \ldots, d_r) \quad \text{where } d_i = d_j \text{ for some } i \text{ and } j\ .$$

☞ We can compute $\mathbf{P}(E)$ by

$$\mathbf{P}(E) = 1 - \mathbf{P}(E^c).$$

E^c is that event that no two dates are the same.

Birthday Problem – continued

☞ The number of outcomes is

$$\left|E^c\right| = (365)(364)(363)\cdots(365-r+1) = \frac{365!}{(365-r)!}.$$

Reason: choose a different date for each person.

So,

$$\mathbf{P}\left(E^c\right) = \frac{\left|E^c\right|}{\left|S\right|} = \frac{365!}{(365-r)!365^r}.$$

☞ So, the probability that at least two people out of r have the same birthday is

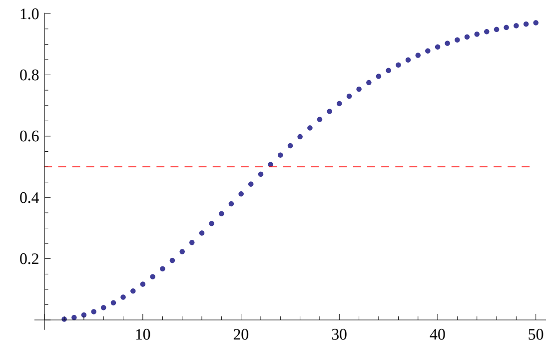
$$\mathbf{P}\left(E\right) = 1 - \mathbf{P}\left(E^c\right) = 1 - \frac{365!}{(365-r)!365^r}.$$

Birthday Problem – resolved

☞ The probability that at least two people out of r have the same birthday is

$$1 - \frac{365!}{(365-r)!365^r}.$$

How big must r be for the value to exceed $\frac{1}{2}$? $r = 23$:



Birthday probabilities for various r

☞ Here are the probabilities of two or more matching birthdays in a group of size r, for various values of r:

r	Probability
2	0.003
10	0.112
20	0.411
22	0.476
23	0.507
30	0.706
40	0.891
50	0.970

Birthmate Problem

Birthmate Problem. What is the least number of strangers whose birthdays you need to ask to have a 50-50 chance of finding someone with the same birthday?

Solution. The chance that a random stranger does NOT have your birthday is

$$\frac{364}{365}.$$

The chance that r people chosen at random do NOT have your birthday is

$$\left(\frac{364}{365}\right)^r.$$

So, the probability that AT LEAST 1 in r randomly chosen strangers has your birthday is

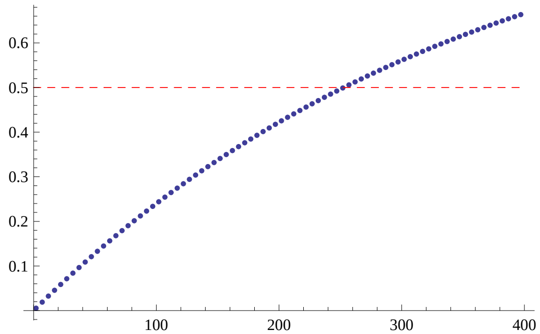
$$1 - \left(\frac{364}{365}\right)^r.$$

Birthmate Problem – resolved

☞ The probability that at one person out of r (randomly chosen) has your same birthday is

$$1 - \left(\frac{364}{365}\right)^r.$$

How big must r be for the value to exceed $\frac{1}{2}$? $r = 253$:



Blackjack: two players

☞ Two cards form a blackjack if

- One card is an ace, and
- the other card is either a ten, a jack, a queen or a king.

Problem. What is the probability of drawing a blackjack when two cards are randomly drawn from a standard deck of 52 cards?

☞ Let the sample space S consist of sets of pairs of cards (order of selection does not matter).

Blackjack: one player

Solution.

☞ There are $\binom{52}{2}$ possible draws of two cards, and each is equally likely (when randomly drawn from a deck of 52 cards).

☞ The number of ways of drawing a blackjack is

$$\binom{4}{1} \cdot \binom{16}{1}.$$

☞ So, the probability of drawing a blackjack is

$$\frac{\binom{4}{1} \cdot \binom{16}{1}}{\binom{52}{2}} = 0.048.$$

Blackjack: two players

Problem. You are playing Blackjack against a dealer. Suppose you have a freshly shuffled deck. What is the probability of each of the following events.

(a) The dealer is dealt a blackjack.

(b) Both of you are dealt a blackjack.

(c) At least one of you is dealt a blackjack.

(d) Neither of you is dealt a blackjack.

(e) Exactly one of you is dealt a blackjack.

Blackjack: two players

Sample space.

☞ Let the sample space S consist of pairs $\{d, y\}$ where d is the set of the dealer's two cards and y is the set of your two cards.
(The order of the cards given out to each of you and the dealer do not matter).

☞ The total number outcomes in the sample space S is

$$\binom{52}{2,2}$$

(We are choosing four cards separated into your cards and the dealer's cards.)

Assumption. All deals are equally likely.

Blackjack: two players

Solution to (a). What is the probability that the dealer is dealt a blackjack?

☞ The number of deals that lead to the dealer getting a blackjack is

$$\binom{4}{1}\cdot\binom{16}{1}\cdot\binom{50}{2}$$

☞ The probability of the dealer being dealt a blackjack is

$$\frac{\binom{4}{1}\cdot\binom{16}{1}\cdot\binom{50}{2}}{\binom{52}{2,2}} \approx 0.0483.$$

Blackjack: two players

Solution to (b). What is the probability that both you and the dealer is dealt a blackjack?

☞ The number of deals that lead to both of you getting a black jack is

$$\left(\binom{4}{1}\cdot\binom{16}{1}\right)\cdot\left(\binom{3}{1}\cdot\binom{15}{1}\right)$$

☞ The probability of the dealer and you being dealt a blackjack is

$$\frac{\binom{4}{1}\cdot\binom{16}{1}\cdot\binom{3}{1}\cdot\binom{15}{1}}{\binom{52}{2,2}} \approx 0.0018.$$

Blackjack: two players

Solution to (c). What is the probability that at least one blackjack is dealt?
☞ Consider the events:

- D: the dealer is dealt a blackjack,
- Y: you are dealt a blackjack.

We want to compute $\mathbf{P}(D \cup Y)$.
☞ Since $\mathbf{P}(D) = \mathbf{P}(Y)$, we have by Proposition 4.3:

$$\begin{aligned}\mathbf{P}(D \cup Y) &= \mathbf{P}(D) + \mathbf{P}(Y) - \mathbf{P}(D \cap Y) \\ &= 2\cdot\mathbf{P}(D) - \mathbf{P}(D \cap Y).\end{aligned}$$

We can compute the right-side from parts (a) and (b).
The probability that at least one of you is dealt a blackjack is

$$\frac{2\cdot\binom{4}{1}\cdot\binom{16}{1}\cdot\binom{50}{2} - \binom{4}{1}\cdot\binom{16}{1}\cdot\binom{3}{1}\cdot\binom{15}{1}}{\binom{52}{2,2}} \approx 0.09476.$$

Blackjack: two players

Solution to (d). What is the probability that neither player is dealt a blackjack?

☞ Let E be the event at least one blackjack is dealt.
So, E^c is the event that no blackjack is dealt.

☞ By Proposition 4.1

$$\mathbf{P}(E^c) = 1 - \mathbf{P}(E),$$

and we computed $\mathbf{P}(E)$ in part (c).
The probability that no blackjack is dealt is

$$1 - \frac{2 \cdot \binom{4}{1} \cdot \binom{16}{1} \cdot \binom{50}{2} - \binom{4}{1} \cdot \binom{16}{1} \cdot \binom{3}{1} \cdot \binom{15}{1}}{\binom{52}{2,2}} \approx 0.9052.$$

Blackjack

Blackjack: two players

Solution to (e). What is the probability that at exactly one blackjack is dealt?

☞ The number of deals that lead to exactly one black jack is

$$2 \cdot \left(\binom{4}{1} \cdot \binom{16}{1}\right) \cdot \left(\binom{3}{2} + \binom{3}{1} \cdot \binom{32}{1} + \binom{15}{2} + \binom{15}{1} \cdot \binom{32}{1} + \binom{32}{2}\right),$$

or $151,040$ possibilities.

☞ The probability of exactly one blackjack is

$$\frac{151040}{\binom{52}{2,2}} \approx 0.0930.$$