Math 425
Introduction to Probability
Lecture 5

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The probability model

☞ The basis of probability is the experiment:

- An experiment is a repeatable procedure that has a a measurable outcome which cannot be predicted ahead of time.

☞ The probability model consists of three ingredients

1. A sample space S of possible outcomes of an experiment,
2. A collection $\mathcal{E}$ of events, which are subsets of S, and are said to occur as the outcome of an experiment.
3. A probability measure $\mathbf{P}$ which assigns a nonnegative real number to events (the "probability of the event occurring as the outcome of an experiment").

Events

☞ Which subsets of the sample space S are events?

Discrete sample space. In a discrete sample space, we can safely take any subset of the sample space S to be an event, which has some probability of occurring.

Nondiscrete sample space. Not all subsets of a nondiscrete sample space can be events (which have some probability of occurring). This issue will **never be a problem for us** in this class. It is actually hard to construct such events in the continuous sample spaces in which we will be most interested.

Enter Tyche

☞ A probability measure on a sample space S is a function $\mathbf{P}$ defined on the events E of the space S. We write

$\mathbf{P}(E)$ to mean the probability of the event E occuring

Sample space. Whenever we talk about a probability measure $\mathbf{P}$, we are assuming it is a measure with respect to some given sample space S.

☞ A probability measure must also satisfy the three axioms which follow.

Axiom 1

Axiom (1 – Nonnegative)

For any event E, $\boldsymbol{P}(E)$ is a real number and

$$0 \leq \boldsymbol{P}(E).$$

☞ This axiom simply states that $\mathbf{P}(E)$ is a nonnegative real number for any event E.

Axiom 2

Axiom (2 – Unit measure)

$$\boldsymbol{P}(S) = 1.$$

☞ The certain event S must always occur, and is defined to have probability 1.

Axiom 3 – Weak form

Axiom (3 – Addition rule (weak form))

For any mutually exclusive events E and F (so, $E \cap F = \emptyset$),

$$\boldsymbol{P}(E \cup F) = \boldsymbol{P}(E) + \boldsymbol{P}(F).$$

☞ This version of Axiom 3 is most relevant when the sample space S is finite.

Compare this axiom to the Sum Rule for counting:

- If two events E and F are mutually exclusive and E has n outcomes and F has m outcomes, then $E \cup F$ has $n + m$ outcomes.

Extended Addition Law

There is nothing special about two events.

Proposition (Extended Addition Law)

If events $E_1, E_2, \ldots, E_n$ are mutually exclusive (so, $E_i \cap E_j = \emptyset$ whenever $i \neq j$), then

$$\boldsymbol{P}(E_1 \cup E_2 \cup \ldots \cup E_n) = \boldsymbol{P}(E_1) + \boldsymbol{P}(E_2) + \ldots + \boldsymbol{P}(E_n).$$

In shorthand,

$$\boldsymbol{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \boldsymbol{P}(E_i).$$

Proof

I'll prove the case of $n = 3$. Let E, F and G be mutually exclusive events. Then $E \cup F$ and G are also mutually exclusive. (Why?)

☞ Apply the Addition Rule (Axiom 3):

$$\begin{aligned} \mathbf{P}(E \cup F \cup G) &= \mathbf{P}((E \cup F) \cup G) \\ &= \mathbf{P}(E \cup F) + \mathbf{P}(G) \\ &= \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \end{aligned}$$

☞ The n-event case is proved by induction, in the same way.

Axiom 3 – Strong form

Axiom (3 – Addition rule (strong form))

For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (so, $E_i \cap E_j = \emptyset$ whenever $i \neq j$),

$$\mathbf{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbf{P}(E_k).$$

☞ This rule is only relevant when the sample space S is infinite. The Strong form of the Addition Rule really is stronger than its Weak form. (See Ross, p. 30).

Probability of the Impossible Event

Recall. $\emptyset$ is the impossible event.

Proposition

$$\mathbf{P}(\emptyset) = 0.$$

Proof.

Both $S = S \cup \emptyset$ and $S \cap \emptyset = \emptyset$.

Apply the Addition Rule (Axiom 2):

$$\mathbf{P}(S) = \mathbf{P}(S \cup \emptyset) = \mathbf{P}(S) + \mathbf{P}(\emptyset).$$

So, $\mathbf{P}(\emptyset) = 0$. □

Proposition 4.1

Proposition (Ross 4.1)

For any event E,

$$\mathbf{P}(E^c) = 1 - \mathbf{P}(E).$$

Proof.

Both $S = E \cup E^c$ and $E \cap E^c = \emptyset$. So,

$$\begin{aligned} 1 &= \mathbf{P}(S) && \text{Axiom 2} \\ &= \mathbf{P}(E \cup E^c) \\ &= \mathbf{P}(E) + \mathbf{P}(E^c) && \text{Axiom 3} \end{aligned}$$

So, $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$. □

Proposition 4.2

Proposition (Ross, 4.2)

Let E and F be any events. If $E \subset F$ then $\mathbf{P}(E) \leq \mathbf{P}(F)$.

As a consequence, $0 \leq \mathbf{P}(E) \leq 1$ for any event E.

Proof of Proposition 4.2

Proof.

Since $E \subseteq F$, we can express F as (see next slide)

$$F = E \cup (E^c \cap F)$$

where E and $E^c \cap F$ are mutually exclusive.

By the Addition Rule

$$\mathbf{P}(F) = \mathbf{P}(E) + \mathbf{P}(E^c \cap F) \geq \mathbf{P}(E),$$

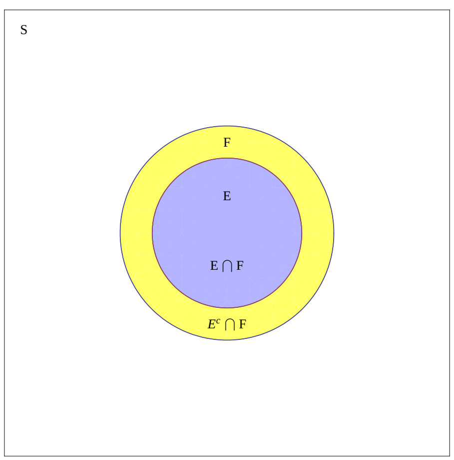
where the last is because $\mathbf{P}(E^c \cap F) \geq 0$ by Axiom 1. □

Partition for Proposition 4.2

Principle. Let E and F be any events in the same sample space.

Then $E = E \cap F$ and $E^c \cap F$ are mutually exclusive, and

$$F = (E \cap F) \cup (E^c \cap F) = E \cup (E^c \cap F).$$



Proposition 4.3

☞ The following is the simplest version of the Inclusion-Exclusion Identity. (See Ross, Proposition 4.4. This is for a later lecture.)

Proposition (Ross 4.3)

Let E and F be any events. Then

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$$

Proof of Proposition 4.3

☞ By the Addition Rule (see next slide)

$$\mathbf{P}(E \cup F) = \mathbf{P}(E \cup F^c) + \mathbf{P}(E^c \cup F) + \mathbf{P}(E \cap F).$$

☞ On the other hand, the Addition Rule gives (see next slide)

$$\mathbf{P}(E) = \mathbf{P}(E \cup F^c) + \mathbf{P}(E \cap F) \qquad \mathbf{P}(F) = \mathbf{P}(E^c \cup F) + \mathbf{P}(E \cap F)$$

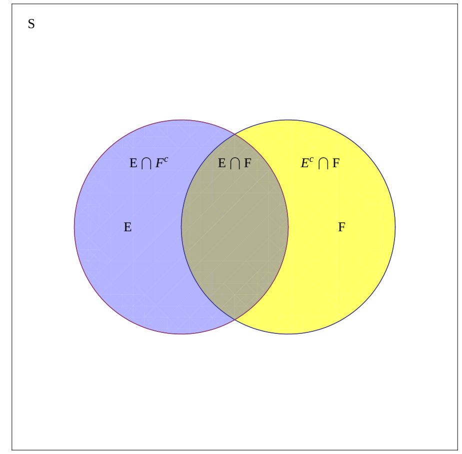
So,

$$\begin{aligned} \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F) &= \left[\mathbf{P}(E \cup F^c) + \mathbf{P}(E^c \cup F) + 2 \cdot \mathbf{P}(E \cap F)\right] \\ &\quad - \mathbf{P}(E \cap F) \\ &= \mathbf{P}(E \cup F^c) + \mathbf{P}(E^c \cup F) + \mathbf{P}(E \cap F) \\ &= \mathbf{P}(E \cup F) \end{aligned}$$

Partition for Proposition 4.3

Principle. Let E and F be any events in the same sample space.
Then each of $E \cap F$, $E^c \cap F$ and $E \cap F^c$ are mutually exclusive, and

$$E \cup F = (E \cap F) \cup (E^c \cap F) \cup (E \cap F^c).$$



Example 1

Example

A retail store accepts VISA and AMEX. It has found that 24 percent of its customers carry AMEX, 61 percent carry VISA and 11 percent carry both.

What percentage of its customers carry VISA or AMEX?

Example 1 – solution

Solution. Consider the events:

- A: customers who carry AMEX,
- V: customers who carry VISA.

Then, from the data

$$\mathbf{P}(A) = 0.24 \quad \mathbf{P}(V) = 0.61 \quad \mathbf{P}(A \cap V) = 0.11$$

Apply Proposition 4.4,

$$\mathbf{P}(A \cup V) = \mathbf{P}(A) + \mathbf{P}(V) - \mathbf{P}(A \cap V) = 0.24 + 0.61 - 0.11 = 0.74.$$

☞ So, 74 percent of the customers carry a credit card.

Example 2

Example

Sixty percent of students at a certain school are neither Greek nor participate in an intramural sports. Twenty percent are Greek and 30 percent participate in intramural sports.

(a) What is the percentage of students who are Greek or participate in intramural sports?

(b) What is the percentage of student who do both?

Example 2 – solution

Solution. Consider the events:

- G: students who are Greek
- I: students involved in intramural sports.

Then, from the data (since $G^c \cap I^c = (G \cup I)^c$)

$$\mathbf{P}\left((G \cup I)^c\right) = 0.6 \quad \mathbf{P}(G) = 0.2 \quad \mathbf{P}(I) = 0.3$$

(a). By Proposition 1.1

$$\mathbf{P}(G \cup I) = 1 - \mathbf{P}\left((G \cup I)^c\right) = 1 - 0.6 = 0.4$$

So, (a) 40 percent of student are either Greek or participate in intramural sports.

(b). Apply Proposition 4.3,

$$\mathbf{P}(G \cap I) = \mathbf{P}(G) + \mathbf{P}(I) - \mathbf{P}(G \cup I) = 0.2 + 0.3 - 0.4 = 0.1$$

So, (b) 10 percent of student participate in both activities.

Finite sample spaces, equiprobable outcomes

☞ Let S is a finite sample space, and suppose it has N outcomes, listed as

$$S = \{a_1, a_2, \ldots, a_N\}$$

☞ Suppose every outcome is equiprobable:

$$\mathbf{P}(\{a_1\}) = \mathbf{P}(\{a_2\}) = \ldots = \mathbf{P}(\{a_N\})$$

☞ Then it follows from the Axioms that:

1. For every $i \leq N$,
$$\mathbf{P}(\{a_i\}) = \frac{1}{N},$$
2. For any event E,
$$\mathbf{P}(E) = \frac{|E|}{N},$$
where $|E|$ is the number of outcomes in E.

Example 1

Example

Assume all outcomes of a throw of a pair of dice are equiprobable. What is the probability of rolling the same value on both die?

Solution. There are 36 possible outcomes on a pair of dice (any number 1 to 6 could appear on each die). The throws with the same face are:

$$(1,1),\ (2,2),\ (3,3),\ (4,4),\ (5,5),\ (6,6).$$

☞ The desired probability is $\frac{6}{36} = \frac{1}{6}$.

Example 2

Example

A coin is tossed five times in succession. Assume each sequence of heads/tails is equiprobable.

What is the probability of getting heads on the first or third toss?

☞ The sample space S for this problem are sequences

$$(a_1, a_2, a_3, a_4, a_5) \quad \text{where each } a_i = H \text{ or } T$$

For example: (H, T, H, T, H).

Example 2 – solution

Solution. There are 2^5 possible outcomes.

☞ Let E_i be the event that the ith toss is heads. Then $|E_i| = 2^4$ (fix ith toss at H). So,

$$\mathbf{P}(E_i) = \frac{2^4}{2^5} = \frac{1}{2}.$$

☞ For $i \neq j$, $|E_i \cap E_j| = 2^3$ (fix ith and j toss at H). So,

$$\mathbf{P}(E_i \cap E_j) = \frac{2^3}{2^5} = \frac{1}{4}.$$

☞ Apply Proposition 4.4:

$$\mathbf{P}(E_1 \cup E_3) = \mathbf{P}(E_1) + \mathbf{P}(E_3) - \mathbf{P}(E_1 \cap E_3) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}.$$

☞ The desired probability is $\frac{3}{4}$.

Example 3

Example

Some states have the Pick-Six Lottery: A person purchases a ticket, and can choose 6 distinct numbers in the set $\{1, 2, 3, \ldots, 49\}$.

Later a Lottery Machine picks 6 distinct numbers at random in the set $\{1, 2, 3, \ldots, 49\}$.

A winning ticket is one which matches the six numbers chosen by the Machine (in any order of selection).

☞ What are the odds of winning with one ticket?

Example 3 – solution

Solution. The sample space of the Pick-Six Lottery are all subsets of $\{1, 2, 3, \ldots, 49\}$ with six elements:

$$\binom{49}{6} \text{ possible outcomes .}$$

☞ By assumption, all possible selections of numbers are equiprobable, so the likelihood that a ticket is a winner is

$$\frac{1}{\binom{49}{6}} = \frac{1}{13,983,816}.$$

☞ You have better odds getting 23 consecutive tosses of heads on a fair coin (where heads/tails are equally likely on each toss).