# Math 425
# Introduction to Probability
# Lecture 4

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## A probability problem

☞ Lets begin by gaming and gambling, where the ♡ of probability lies.

**Problem**. Which event is more likely?

1. At least one six on 4 throws of a die.
2. At least one pair of sixes on 24 throws of a pair of dice.

**Note**. This problem, posed by the Chevalier de Méré to Blaise Pascal in the seventeenth century led to the modern development of probability.





## An experimental solution

☞ One way to solve this problem is to run two experiments.

1. Throw a die 4 times repeatedly and compute the ratio:

$$p = \frac{\text{number of times at least 1 six occurs in four throws}}{\text{total number of times attempted 4 throws}}$$

2. Throw a pair of dice 24 times and compute the ratio

$$q = \frac{\text{number of times at least 1 double six appears in 24 throws}}{\text{total number times attempted 24 throws}}$$

If $p > q$ the first event is more likely and if $q < p$ the second event is more likely.

**Question**. What problems do you see with this method?





## A mathematical solution

☞ The modern approach to probability is to construct a mathematical model, which is an abstraction of the real-life situation.

☞ You can compute the likelihoods in the idealized model with the necessary precision to distinguish these two probabilities.

☞ If you are clever and lucky in your in creating your abstraction, the mathematical relationships in the model are accurate reflections of reality.

## Experiments

☞ Probability computes the likelihood of an event before it occurs.
Like the real-world, it requires experiments.

**Definition.** An experiment is a repeatable procedure that has a measurable outcome which cannot be predicted ahead of time.
Sometimes the execution of an experiment is called a trial.

**Example.** An experiment.

- Toss a coin three times and record the side of the upturned face (heads or tails.)

The outcomes of this experiment will consist triples:

$$ijk \quad \text{where } i, j, k = H, T$$

(For example, *HHH* for three heads in three tosses.)





## Sample space

The outcomes of an experiment is an abstraction from real-world outcomes.

**Definition.** The sample space $S$ of an experiment is the set of all possible outcomes that could be measured from the experiment.
We will denote the sample space by $S$.

**Example.** The sample space $S$ for the coin tossing experiment.

- Toss a coin three times and record the side of the upturned face.

$$S = \{HHH, HTH, HTT, HHT, THH, THT, TTH, TTT\}.$$

☞ The sample space is an abstraction of real-world outcomes.
These outcomes do not record the time of day, my mood, or the date on the penny. (☺ Although, these could be relevant.)





## Events

☞ Probability measures the uncertainty of events.

**Definition.** An event is a subset of the sample space of an experiment.

**Examples.** The experiment is the tossing of a coin three times.

- The following is the event of tossing a head on the first throw:

$$E = \{HHH, HTH, HTT, HHT\}$$

- The following is the event of tossing two tails:

$$F = \{HTT, THT, TTH\}$$





## Events

**Definition.** If the outcome of an experiment is member of an event, we say the event has occurred.

**Example.** The following are two events in the sample space of tossing a coin three times:

$$E = \{HHH, HTH, HTT, HHT\} \quad F = \{HTT, THT, TTH\}$$

If the outcome of an experiment is *HTH* then we say that:

- $E$ occurs (i.e. the first toss was heads),
- $F$ does not occur (i.e. there were not two tails tossed).

## Events

**Notation.** We will use uppercase letters for events, $E$, $F$ and $G$, as well as subscripts, $E_1$, $F_2$, $G_3$.

☞ Two events for any sample space are

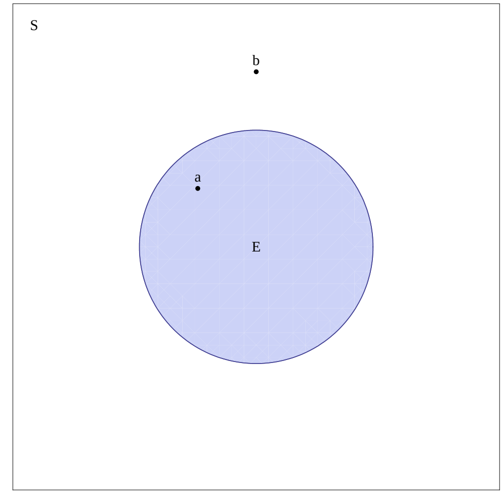
- $S$: the certain event (“something occurs"),
- $\emptyset$: the impossible event ( “nothing occurs").





## Venn Diagrams

☞ An event $E$ in a sample space $S$ can be represented pictorially as a Venn diagram. The outcomes are designated points, and the event is shaded.








## Operations on sets

☞ There are three basic operations on events.

**Definition.** Let $E$ and $F$ be events from a sample space $S$.

- $E \cup F$ is the event $E$ occurs or $F$ occurs,
  This is the union operation.
- $E \cap F$ is the event $E$ occurs and $F$ occurs,
  This is the intersection operation.
- $E^c$ is the event $E$ does not occur,
  This is the complement operation.

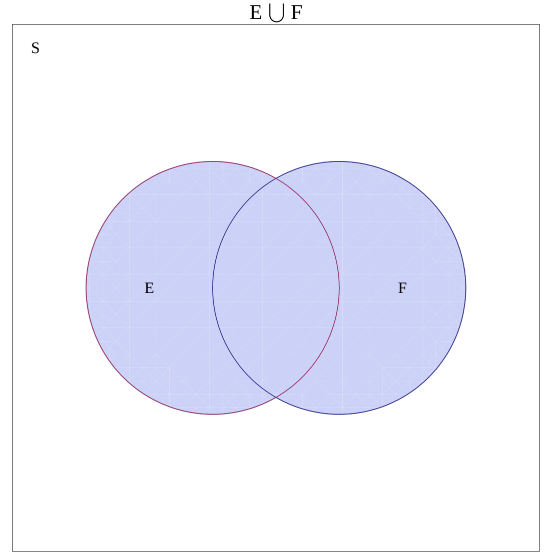
**Note.** Ross writes $EF$ for $E \cap F$.



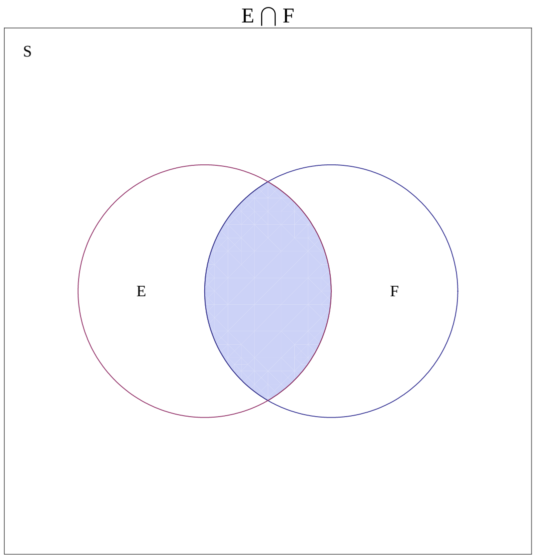


## Unions

$E \bigcup F$



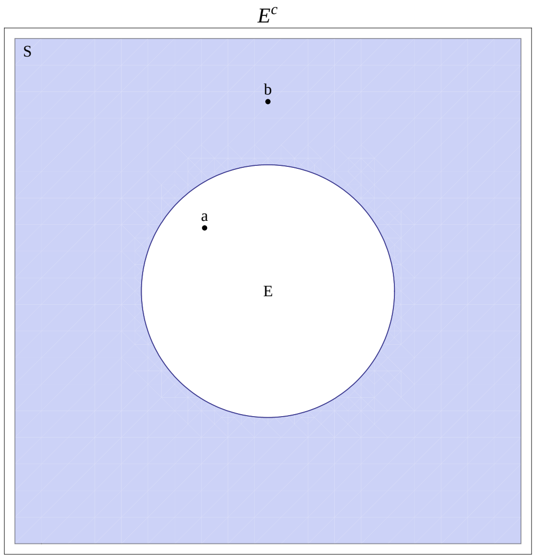
## Intersections








## Complements








## Example

☞ Let $S$ be the space of outcomes of 3 coin tosses. Here are some events:

$$\begin{aligned} E_1 &= \{HHH, HTH, HTT, HTH\} \\ E_2 &= \{HHH, HHT, THH, THT,\} \\ E_3 &= \{HHH, THH, HTH, TTH\} \\ F &= \{TTT\} \end{aligned}$$

☞ We can construct new events from these:

- $E_1 \cup E_2 \cup E_3$ (at least one heads is tossed)
- $F^c$ (at least one heads appears)
- $E_1 \cup E_2$ (a heads appears in the first or second toss)
- $E_1 \cap E_2$ (a heads appears in both the first and second toss)
- $E_1 \cap E_2 \cap E_3$ (a heads appears in all three tosses)





## Mutually exclusive events

**Definition.** Let $E$ and $F$ be events in the same sample space.
$E$ and $F$ are mutually exclusive (or disjoint) if it is impossible for $E$ and $F$ to occur together. That is,

$$E \cap F = \emptyset.$$

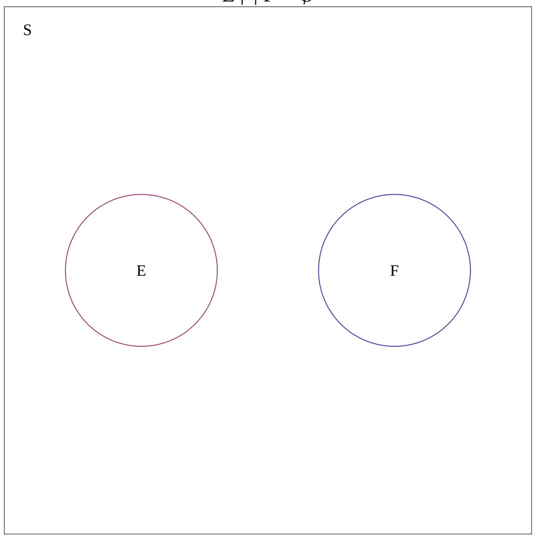
**Example.** Consider the experiment with three tosses of a coin.
Let $E$ and $F$ be the events

$$E = \{HHH, HHT\} \quad F = \{TTH, TTT\} \qquad \text{so, } E \cap F = \emptyset.$$

$E$ and $F$ are mutually exclusive.

## Mutually exclusive events

$E \cap F = \emptyset$








## Subset relation

**Definition.** Let $E$ and $F$ be events in the same sample space.
$E$ is contained in $F$, written $E \subset F$, if whenever $E$ occurs, $F$ occurs as well. (That is, any outcome in $E$ is already an outcome in $F$.)

**Example.** Consider the experiment with three tosses of a coin.
Let $E$ and $F$ be the events

$$E = \{HHH, HHT\} \quad F = \{HHH, HHT, HTH, HTT\}.$$

So, $E \subset F$.

**Note.** For any event $E$,

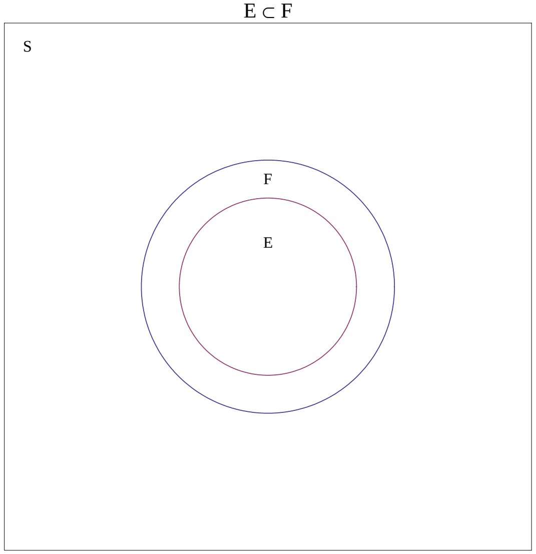
$$\emptyset \subset E \subset S.$$





## Subset relation

$E \subset F$








## Event equality

**Definition.** Let $E$ and $F$ be events in the same sample space.
$E$ and $F$ are the same event, written $E = F$, when $E$ and $F$ always occur together. (That is, they contain the same outcomes.)

$$E = F \quad \text{is equivalent to} \quad E \subset F \text{ and } F \subset E.$$

**Example.** Consider the experiment with three tosses of a coin.

- $E$: the event of getting tails on each toss,
- $F$: the event of getting no heads.

$$E = F = \{TTT\}$$

## Generalizing Notation

It makes sense to take unions and intersection of several events.

**Definition.** Let $E_1, E_2, \ldots, E_n$ be events in a common sample space.

- We write

$$\bigcup_{i=1}^{n} E_i \quad \text{for the event:} \quad \text{at least one of the events } E_i \text{ occurs.}$$

- We write

$$\bigcap_{i=1}^{n} E_i \quad \text{for the event:} \quad \text{each of the events } E_i \text{ occur.}$$

**Note.** Later we will be interested in unions and intersections over infinitely many events $E_1, E_2, \ldots$. We write these as

$$\bigcup_{i=1}^{\infty} E_i \qquad \bigcap_{i=1}^{\infty} E_i.$$





## Summary of events

| **Mathematical events** | **Real-world interpretation** |
|---|---|
| $S$, entire sample space | The certain event 'something happens' |
| The empty subset $\emptyset$ of $S$ | The impossible event 'nothing happens' |
| The intersection $E \cap F$ | 'Both $E$ and $F$ occur' |
| $\bigcap_{i=1}^{n} E_i$ | 'Each of the events $E_1, E_2, \ldots, E_n$ occur' |
| The union $E \cup F$ | 'At least one of $E$ and $F$ occurs' |
| $\bigcup_{i=1}^{n} E_i$ | 'At least one of $E_1, E_2, \ldots, E_n$ occurs' |
| Complement $E^c$ of $E$ | '$E$ does not occur' |
| $E \subseteq F$ | 'If $E$ occurs, then $F$ occurs' |





## Basic Laws of Events

☞ Let $E$, $F$ and $G$ be events in the same sample space.

- Commutativity

$$E \cup F = F \cup E \qquad E \cap F = F \cap E$$

- Associativity

$$(E \cup F) \cup G = E \cup (F \cup G) \qquad (E \cap F) \cap G = E \cap (F \cap G)$$

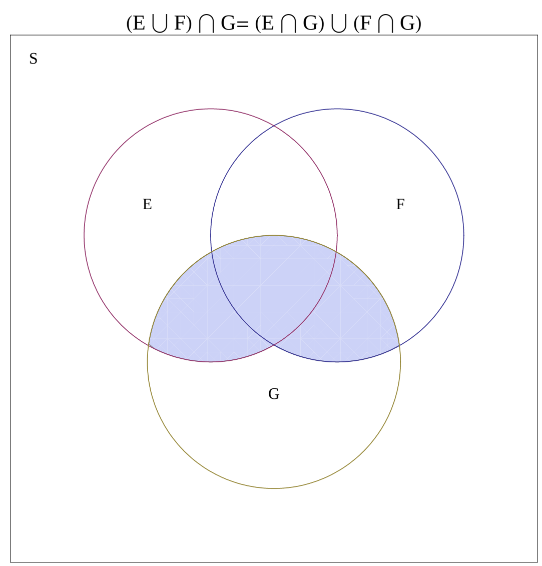
- Distributivity

$$(E \cup F) \cap G = (E \cup G) \cap (F \cup G) \qquad (E \cap F) \cup G = (E \cap G) \cup (F \cap G)$$

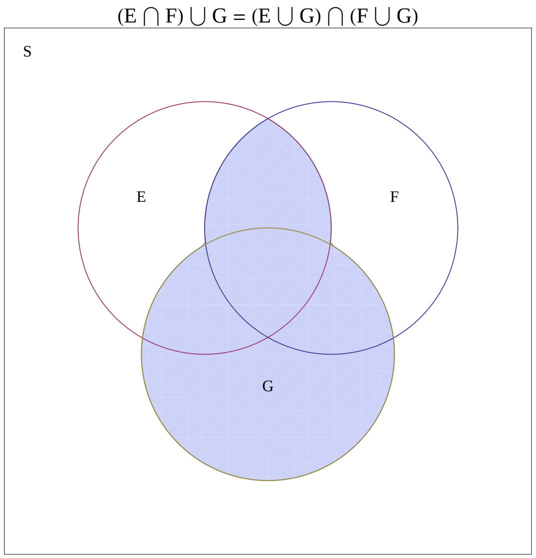




## Distributivity I



# Distributivity II



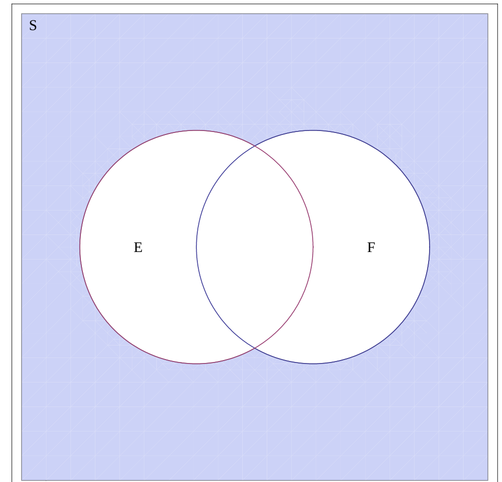





# DeMorgan's Law I

**DeMorgan's Law I**. Let $E$ and $F$ be events in the same sample space. Then

$$(E \cup F)^c \quad = \quad E^c \cap F^c$$



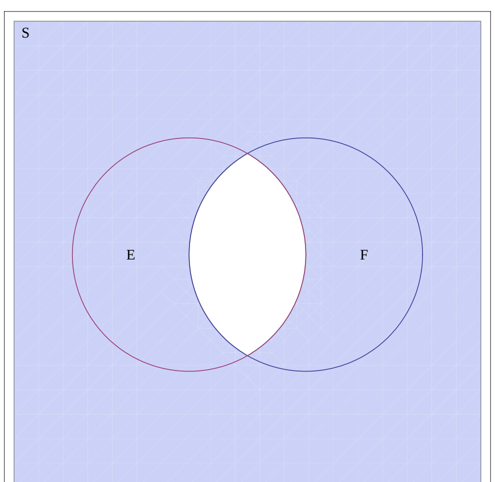





# DeMorgan's Law II

**DeMorgan's Law II**. Let $E$ and $F$ be events in the same sample space. Then

$$(E \cap F)^c \quad = \quad E^c \cup F^c$$



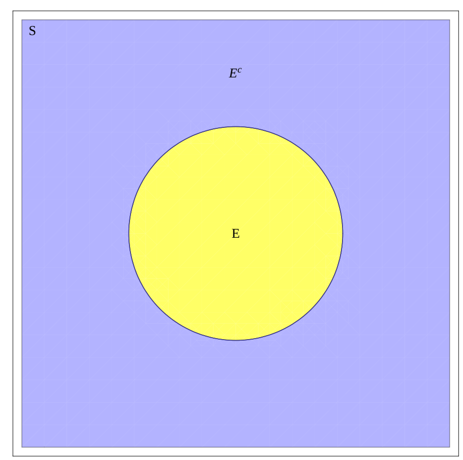





# Partitioning the sample space

**Principle**. Let $E$ be any even in the same sample space. Then $E$ and $E^c$ are mutually exclusive, and

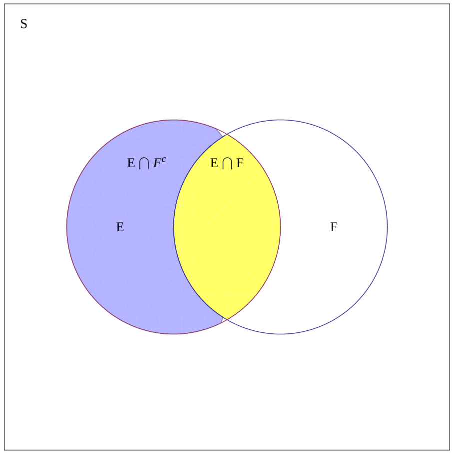
$$S = E \cup E^c.$$



## Partitioning a set

**Principle**. Let $E$ and $F$ be any events in the same sample space. Then $E \cap F$ and $E \cap F^c$ are mutually exclusive, and

$$E = (E \cap F) \cup (E \cap F^c).$$



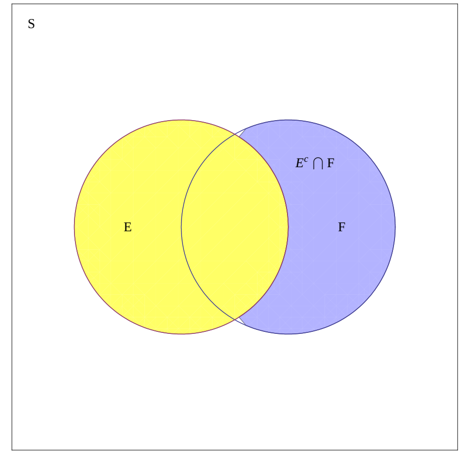





## Partitioning a union

**Principle**. Let $E$ and $F$ be any events in the same sample space. Then $E$ and $E^c \cap F$ are mutually exclusive, and

$$E \cup F = E \cup (E^c \cap F).$$



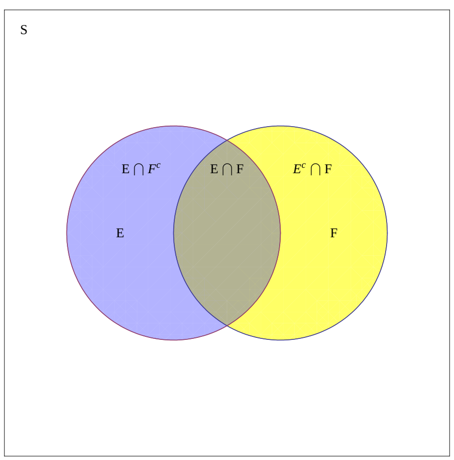





## Partitioning a union

**Principle**. Let $E$ and $F$ be any events in the same sample space. Then $E \cap F^c$, $E \cap F$, and $E^c \cap F$ are mutually exclusive, and

$$E \cup F = (E \cap F^c) \cup (E \cap F) \cup (E^c \cap F).$$








## Three types of sample spaces

☞ There are three kinds of sample spaces

1. Discrete finite sample spaces,
2. Discrete infinite sample spaces,
3. Nondiscrete sample spaces

**Definition**. A sample space is discrete if its outcomes can be enumerated by the natural numbers:

$$S = \{a_1, a_2, a_3, \ldots\}$$

A sample space is finite if its outcomes can be enumerated up to a natural number:

$$S = \{a_1, a_2, a_3, \ldots, a_n\},$$

and it is infinite otherwise.

## Finite sample spaces

☞ Most of our examples in Chapters 2 and 3 will be finite sample spaces (which are discrete).

**Examples.**

- The space of outcomes for a coin tossed 3 times (recording heads/tails).
  There are $2^3 = 8$ outcomes.
- The space of outcomes for a single die thrown 4 times (recording 1 to 6).
  There are $6^4 = 1296$ outcomes.
- The space of a pair for dice thrown 24 times (recording pairs of 1 to 6).
  There are $6^{24}$ outcomes.
- The space of outcomes for the birthdates of 40 people (recording the month and day for each person).
  There are $365^{40}$ outcomes.





## Discrete infinite sample spaces

☞ Discrete infinite sample spaces will increasingly be the object of our study from Chapter 4 onward.

**Examples.**

- The space of outcomes for tosses of a coin until a first heads appears:

$$S = \{H, TH, TTH, TTTH, TTTTH, TTTTTH, \dots\}.$$

- The space of outcomes for throws of a die until a six appears. We might represent this using finite sequences of values 1 to 6:

$$S = \{(6), (1,6), (3,1,6), (2,1,1,6), \dots\}.$$





## Discrete infinite sample spaces

☞ Nondiscrete infinite sample spaces will be the object of our study from Chapter 5 onward. These sample spaces involve continuous probability distributions.

**Examples.** The most common examples will have real valued outcomes.

- The space of outcomes for an ideal random number generator. This space of outcomes is the set of all real numbers between 0 and 1.
- The space of outcomes of the times an $\alpha$ particle is emitted from a collection of Uranium 238 atoms. We might use all nonnegative reals for this space of outcomes (where time 0 is the starting point of our data collection.)
- The space of outcomes for infinite sequences of a coin tosses. These outcomes may be recorded such as

$$HTHTTHHHTTHH\dots$$