Math 425
Intro to Probability
Lecture 30

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Functions of a Random Variable: Density

☞ Let $g(x) = y$ be a one-to-one function whose derivative is nonzero on some region A of the real line.
Suppose g maps A onto B, so that there is an inverse map $x = h(y)$ from B back to A.

☞ Let X be a continuous random variable with known density $f_X(x)$.
Let $Y = G(X)$. Then the density of Y is

$$f_Y(y) = f_X\big(h(y)\big) \cdot \left|\frac{d}{dt}h(y)\right|.$$

Note: Compare to Ross, Theorem 5.7.1, page 243.

Problem

☞ Let the continuous random variables (X, Y) have joint density $f_{X,Y}(x, y)$ and let $\mathcal{A} = \{(x, y) : f_{X,Y}(x, y) > 0\}$.
(X, Y) determines a point with xy-coordinates in the region $\mathcal{A}$.

☞ Consider the continuous random variables (U, V) given by

$$U = g_1(X, Y) \qquad V = g_2(X, Y).$$

(U, V) determines a point with uv-coordinates in some region $\mathcal{B}$.

Problem. If the transformation from xy-coordinates to uv-coordinates given by

$$u = g_1(x, y) \qquad v = g_2(x, y).$$

is nice on $\mathcal{A}$, then we can produce the joint density $f_{U,V}(u, v)$ for the random variable (U, V).

Definition Nice Transformations

Definition

A transformation from xy-coordinates to uv-coordinates ($xy \Rightarrow uv$) given by

$$u = g_1(x, y) \qquad v = g_2(x, y).$$

is nice on $\mathcal{A}$, if

1. The partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ exist and are continuous on $\mathcal{A}$.
2. The Jacobian of the transformation is nonzero on $\mathcal{A}$:

$$J(x, y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0$$

whenever $(x, y) \in \mathcal{A}$.

Change of coordinates

☞ A nice transformation on $\mathcal{A}$ ($xy \Rightarrow uv$) amounts to simply a change of coordinates of the plane from xy-coordinates to uv-coordinates.
We can recover the original xy-coordinates from the new uv-coordinates.

☞ Suppose ($xy \Rightarrow uv$) is nice transformation on $\mathcal{A}$

$$u = g_1(x, y) \qquad v = g_2(x, y)$$

to uv-coordinates on a region $\mathcal{B}$.
There is a reverse transformation ($uv \Rightarrow xy$) from uv-coordinates to xy-coordinates

$$x = h_1(u, v) \qquad y = h_2(u, v).$$

which maps $\mathcal{B}$ onto $\mathcal{A}$ and which are also nice on $\mathcal{B}$.

Jacobians

☞ The Jacobian of the original transformation ($xy \Rightarrow uv$) is the determinant

$$J(x, y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

☞ The Jacobian of the inverse transformation ($uv \Rightarrow xy$) is the determinant

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

☞ Since ($xy \Rightarrow uv$) is nice on $\mathcal{A}$,
$J(x, y) \neq 0$ whenever $(x, y) \in \mathcal{A}$ and
$J(u, v) \neq 0$ whenever $(u, v) \in \mathcal{B}$.

Furthermore, the two Jacobian determinants are inverses

$$J(x, y) = J(u, v)^{-1}$$

Main Theorem

Theorem

Let (X, Y) be continuous random variables with joint density $f_{X,Y}(x, y)$, and (U, V) be random variables given by

$$U = g_1(X, Y) \qquad V = g_2(X, Y).$$

Suppose the ($xy \Rightarrow uv$) transformation

$$u = g_1(x, y) \qquad v = g_2(x, y).$$

is nice on $\mathcal{A} = \{(x, y) : f_{X,Y}(x, y) \neq 0\}$.
Let the inverse ($uv \Rightarrow xy$) from $\mathcal{B}$ to $\mathcal{A}$ be

$$x = h_1(u, v) \qquad y = h_2(u, v).$$

☞ *The joint density of (U, V) is given for $(u, v) \in \mathcal{B}$ by either equation*

$$f_{U,V}(u, v) = f_{X,Y}\big(h_1(u, v), h_2(u, v)\big) \cdot \big|J(u, v)\big|$$

$$f_{U,V}(u, v) = f_{X,Y}\big(h_1(u, v), h_2(u, v)\big) \cdot \big|J(x, y)^{-1}\big|$$

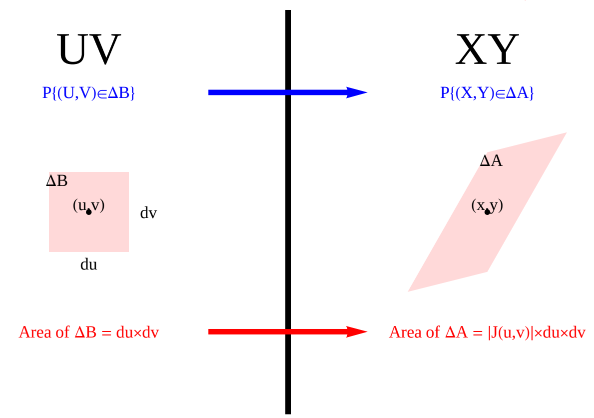
whichever is more convenient to compute.

Picture of Theorem

$$f_{U,V}(u, v) \cdot du \cdot dv \approx \mathbf{P}\{(U, V) \in \Delta B\}$$

$$\|$$

$$\mathbf{P}\{(X, Y) \in \Delta A\} \approx f_{X,Y}(x, y) \cdot \big|J(u, v)\big| \cdot du \cdot dv$$



Sketch of Proof of Theorem

☞ Let $B \subseteq \mathcal{B}$ and suppose $(uv \Rightarrow xy)$ maps B to $A \subseteq \mathcal{A}$.

$$\begin{aligned} \mathbf{P}\{(U,V) \in B\} &= \mathbf{P}\{(X,Y) \in A\} \\ &= \int\int_{(x,y)\in A} f_{X,Y}(x,y)\,dx\,dy \\ &= \int\int_{(u,v)\in B} f_{X,Y}(h_1(u,v),h_2(u,v)) \cdot |J(u,v)|\,du\,dv \end{aligned}$$

using the Change of Variables Theorem of analysis.

☞ Intuitively, we can break B into small regions ΔB which $(uv \Rightarrow xy)$ transforms to small regions ΔA of $\mathcal{A}$ where for any $(u,v) \in \Delta B$:

$$f_{U,V}(u,v) \cdot \text{Area}(\Delta B) \approx f_{X,Y}(h_1(u,v),h_2(u,v)) \cdot \text{Area}(\Delta A)$$

where $\text{Area}(\Delta A) \approx \text{Area}(\Delta B) \cdot |J(u,v)|$.

☞ Differentiate the integrals to get the transformation rule:

$$f_{U,V}(u,v) = \begin{cases} f_{X,Y}(h_1(u,v),h_2(u,v)) \cdot |J(u,v)| & \text{if } (u,v) \in \mathcal{B} \\ 0 & \text{otherwise.} \end{cases}$$

Functions of a Random Variable: Density

Example. Let X and Y be continuous random variables with joint density $f_{X,Y}(x,y)$ and where $X \neq 0$. Consider

$$U = XY \qquad V = X.$$

☞ The transformation $(xy \Rightarrow uv)$ is given by

$$u = xy \qquad v = x.$$

The inverse transformation $(uv \Rightarrow xy)$ is given by

$$x = v \qquad y = \frac{u}{v}.$$

The Jacobian for transformation for $(uv \Rightarrow xy)$ is

$$J(u,v) = \begin{vmatrix} 0 & \frac{1}{v} \\ 1 & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}$$

Functions of a Random Variable: Density

☞ So, the joint density is

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(h_1(u,v),h_2(u,v)) \cdot |J(u,v)| \\ &= f_{X,Y}\left(v, \frac{u}{v}\right) \cdot \frac{1}{|v|} \end{aligned}$$

☞ We can compute the marginal $f_U(u) = f_{XY}(u)$ by

$$f_{XY}(u) = \int_{-\infty}^{\infty} f_{X,Y}\left(v, \frac{u}{v}\right) \cdot \frac{1}{|v|}\,dv$$

Rectangle to Polar coordinates

☞ It is often convenient to change from rectangular coordinates xy to polar coordinates $r\theta$. The transformation $(xy \Rightarrow r\theta)$ is

$$r = x^2 + y^2 \qquad \theta = \arctan\frac{y}{x}.$$

where $r > 0$ and $\quad -\pi < \theta \leq \pi$.

☞ The inverse transformation $(r\theta \Rightarrow xy)$ from polar back to rectangular is

$$x = r\cos\theta \qquad y = r\sin\theta.$$

☞ The transformation is $(xy \Rightarrow r\theta)$ nice in the punctured plane $\mathbb{R}^2 - \{(0,0)\}$. Verified in three slides.

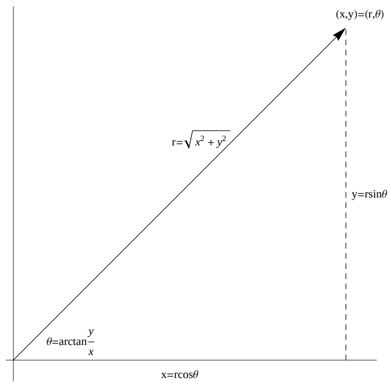
Converting Rectangle to Polar

☞ Rectangle xy-coordinates to polar $r\theta$-coordinates:

$$r = \sqrt{x^2 + y^2}, \quad r > 0 \qquad \theta = \arctan\frac{y}{x}, \quad -\pi < \theta \le \pi,$$

Polar $r\theta$-coordinates to rectangle xy-coordinates

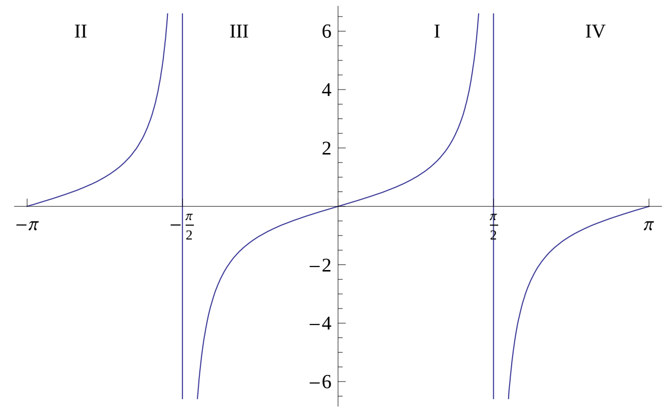
$$x = r\cos\theta \qquad y = r\sin\theta \quad -\infty < x, y < \infty.$$



Converting Rectangle to Polar

☞ Plot of $\tan\frac{y}{x}$ on $[-\pi, \pi]$. The four quadrants of the plane are

$$I : x, y > 0 \quad II : x < 0, y > 0 \quad III : x, y < 0 \quad IV : x > 0, y < 0$$



Problem: Rectangle to Polar

Problem. Let (X, Y) be randomly chosen in some region R of the xy-plane with joint density $f_{X,Y}(x, y)$.

☞ Consider the random variables giving the polar coordinates

$$R = \sqrt{X^2 + Y^2} \qquad \Theta = \arctan\frac{Y}{X}$$

where $R > 0$ and $-\pi < \Theta \le \pi$.

☞ The Jacobian is easiest to compute on the $r\theta$-plane:

$$J(r, \theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

☞ The joint distribution of R, Θ is

$$f_{R,\Theta}(r, \theta) = r \cdot f_{X,Y}(r\cos\theta, r\sin\theta) \qquad r > 0, -\pi < \theta \le \pi.$$

Example

Example. Let (X, Y) be uniformly distributed in $R =$ unit circle. So,

$$f_{X,Y}(x, y) = \frac{1}{\pi} \qquad \text{when } x^2 + y^2 \le 1.$$

☞ So,

$$f_{R,\Theta}(r, \theta) = r \cdot f_{X,Y}(r\cos\theta, r\sin\theta) = \frac{r}{\pi} \qquad 0 < r \le 1, -\pi < \theta \le \pi$$

☞ The marginals are

$$\begin{aligned} f_R(r) &= \int_{-\pi}^{\pi} \frac{r}{\pi}\, d\theta = 2r \qquad 0 < r \le 1, \\ f_\Theta(\theta) &= \int_0^1 \frac{r}{\pi}\, dr = \frac{1}{2\pi} \qquad -\pi < \theta \le \pi. \end{aligned}$$

Thus, Θ is uniformly distributed on $(-\pi, \pi]$.

Example

Example. Let (X, Y) be independent and normally distributed in the plane with $(\mu = 0, \sigma^2)$. So,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

☞ Since $f_{R,\Theta}(r,\theta) = r \cdot f_{X,Y}(r\cos\theta, r\sin\theta)$,

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)/2\sigma^2} = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \qquad 0 < r, -\pi < \theta \le \pi.$$

☞ The marginals are

$$\begin{aligned} f_R(r) &= \int_{-\pi}^{\pi} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\, d\theta = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \qquad 0 < r, \\ f_\Theta(\theta) &= \int_0^{\infty} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\, dr = \frac{1}{2\pi} \qquad -\pi < \theta \le \pi. \end{aligned}$$

Thus, Θ is uniformly distributed on $(-\pi, \pi]$ and R is the Rayleigh distribution (the distance of (X, Y) from the origin).

Example

Example. Let R be exponentially distributed with mean 2 and Θ be uniformly distributed in $(-\pi, \pi]$, both independent. The joint distribution is

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi} \cdot \frac{1}{2} e^{-r/2} \qquad 0 < r, -\pi < \theta \le \pi$$

☞ Let X and Y be random variables determined by

$$X = \sqrt{R}\cos\Theta \qquad Y = \sqrt{R}\sin\Theta$$

Solve for r, θ in the transformation $x = \sqrt{2r}\cos\theta$ and $y = \sqrt{2r}\sin\theta$:

$$r = x^2 + y^2 \qquad \theta = \arctan\frac{y}{x}.$$

The Jacobian determinant is easiest to compute using $r\theta$-coordinates:

$$J(r,\theta) = \begin{vmatrix} \frac{\cos\theta}{2\sqrt{r}} & \frac{\sin\theta}{2\sqrt{r}} \\ -\sqrt{r}\sin\theta & \sqrt{r}\cos\theta \end{vmatrix} = \frac{\cos^2\theta}{2} + \frac{\sin^2\theta}{2} = \frac{1}{2}.$$

Example – continued

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi} \cdot \frac{1}{2} e^{-r/2} \qquad 0 < r, -\pi < \theta \le \pi$$

☞ So,

$$\begin{aligned} f_{X,Y}(x,y) &= f_{R,\theta}(x^2+y^2, \arctan\frac{y}{x}) \cdot 2 \\ &= \frac{1}{2\pi} e^{-(x^2+y^2)/2} \end{aligned}$$

☞ X and Y are independent and normally distributed random variables with $(\mu = 0, \sigma^2 = 1)$. The marginals are obtained by integrating $f_{X,Y}(x,y)$:

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ f_Y(y) &= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \end{aligned}$$

Example–continued

Let U and V be uniformly distributed on $(0, 1)$.

☞ Consider the random variable Θ:

$$\Theta = 2\pi V - \pi$$

So, Θ is uniformly distributed on $(-\pi, \pi)$.

☞ Consider the random variable R:

$$R = 2\ln\frac{1}{U} \qquad \text{solving, } u = e^{-r/2}.$$

By Proposition 5.7.1 (Ross, page 243),

$$f_R(r) = f_U(e^{-r/2}) \cdot |u'| = \frac{1}{2} e^{-r/2}$$

So, R is exponentially distributed with mean 2.

Example – continued

☞ Let X and Y be random variables determined by

$$X = \sqrt{R}\cos\Theta \qquad Y = \sqrt{R}\sin\Theta$$

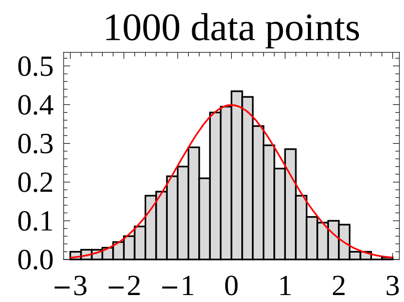
Then X and Y are independent standard normal random variables!!

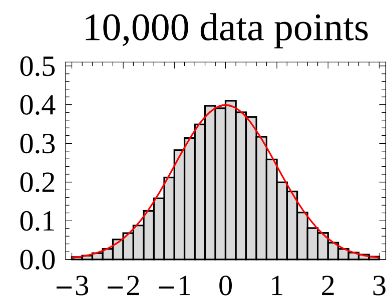
☞ We can simulate a standard normal random variable X by using two independent uniform random variables U and V on $(0,1)$:

$$X = \sqrt{2\ln\frac{1}{U}}\cos\left(2\pi V - \pi\right).$$

Converting Rectangle to Polar

☞ Simulating a standard normal random variable with a pair of independent uniform random variables on $(0,1)$.





Example

Example. Let X and Y be independent and uniformly distributed on $(0,1]$. Find the joint probability density function for the random variables

$$U = \frac{X}{Y} \qquad V = XY.$$

☞ Individually, the distribution of X and Y are

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

So, the joint distribution $f_{X,Y}(x,y)$ is

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \begin{cases} 1 & \text{if } 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example – continued

☞ The transformation into uv-coordinates

$$u = \frac{x}{y} \qquad v = xy,$$

is one-to-one and has an inverse

$$x = \sqrt{uv} \qquad y = \sqrt{\frac{v}{u}}.$$

The Jacobian determinant is easiest when computed in xy coordinates:

$$J(x,y) = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ y & x \end{vmatrix} = 2\frac{x}{y} = 2u$$

☞ So, $u, v > 0$ and

$$f_{U,V}(u,v) = f_{X,Y}\left(\sqrt{uv}, \sqrt{\frac{v}{u}}\right)\cdot\left|\frac{1}{2u}\right| = \begin{cases} \frac{1}{2u} & \text{if } 0 < \sqrt{uv}, \sqrt{\frac{v}{u}} \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example – continued

☞ It remains to compute the bounds on u and v.

$$0 < \sqrt{uv}, \sqrt{\frac{v}{u}} \le 1 \implies 0 < v \le \frac{1}{u},\ 0 < v \le u.$$

Only one of these ranges need be retained, depending upon whether $u \in (0,1]$ or $u \in [1,\infty)$:

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{2u} & \text{if } 0 < u < 1,\ 0 < v \le u, \\ & \text{or, if } u \ge 1,\ 0 < v \le \frac{1}{u}, \\ 0 & \text{otherwise.} \end{cases}$$

Example – continued

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{2u} & \text{if } 0 < u < 1,\ 0 < v \le u, \\ & \text{or, if } u \ge 1,\ 0 < v \le \frac{1}{u}, \\ 0 & \text{otherwise.} \end{cases}$$

☞ We compute the marginals.

$$f_U(u) = \begin{cases} \int_0^u \frac{1}{2u}\,dv & \text{if } 0 < u < 1 \\ \int_0^{\frac{1}{u}} \frac{1}{2u}\,dv & \text{if } u \ge 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}\,dv & \text{if } 0 < u < 1 \\ \frac{1}{2u^2}\,dv & \text{if } u \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_V(v) = \begin{cases} \int_v^{\frac{1}{v}} \frac{1}{2u}\,du & \text{if } 0 < v \le 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \ln\frac{1}{v} & \text{if } 0 < v \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Example – continued

☞ Plot of area determined by

$$0 < u < 1 \implies 0 < v \le u \quad \text{and} \quad u \ge 1 \implies 0 < v \le \frac{1}{u}.$$