# Math 425
# Introduction to Probability
# Lecture 3

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January 12, 2009





## Problem: Sitting in a circle

**Problem.** 5 boys and 5 girls are to be assigned seats so that boys and girls alternate. How many arrangements are there if they are sat in a line? How many if they are sat in a circle?

**Solution.**

1. If they are sat in a line, there are two arrangements of gender:

   $$G\,B\,G\,B\,G\,B\,G\,B\,G\,G \qquad B\,G\,B\,G\,B\,G\,B\,G\,B\,G$$

   and each arrangement has $5! \cdot 5!$ possibilities. The number of arrangements is $2 \cdot 5! \cdot 5! = 28,800$.
2. If they are sat in a circle?, then there is only one arrangement of gender, and this has $5! \cdot 5! = 14,400$ possible arrangements of people.





## Problem: Sitting in a circle

**Problem.** $N$ people to sit in a circle. How many different ways are there seating people next to each other?

**Solution.**

1. Fix one point on the circle and number the seats clockwise. There are $N!$ ways to assign people to seats.
2. Fix one arrangement. If we rotate the circle by 1 seat, we have the same arrangement of people (although they are now shifted over one seat). So, we have over counted by $N$ is step 1.

There are $\frac{N!}{N} = (N-1)!$ ways to assign $N$ people seats in a circle, where we are only concerned with who is sitting next to whom.





## Binomial theorem

The values $\binom{n}{r}$ are called the binomial coefficients because of the following theorem.

**Theorem (Binomial Theorem)**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

## Proof of the Binomial Theorem

☞ Consider the product as $n$ factors of $(x+y)$.

$$(x+y)^n = (x+y)(x+y)\cdots(x+y).$$

☞ How many ways are there of generating $x^k y^{n-k}$ for a fixed $k$?

Choose $x$ from $k$ out of $n$ factors $(x+y)$ (and $y$ from the remaining $n-k$ factors): $\binom{n}{k}$ ways.

So, the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$.

☞ Thus,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$





## Proposition

**Proposition.** (See Example 4E).

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

**Proof 1– Combinatorial argument.** A subset must have a size $\leq n$.

- For each $k \leq n$, there are $\binom{n}{k}$ subsets of size $k$.

The number of subsets (using the Sum Rule) is

$$\sum_{k=0}^{n} \binom{n}{k}.$$

But, there are $2^n$ subsets:

- For each element, choose IN or OUT – there are $2^n$ possible outcomes, each outcome is a different subset.

So,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$





## Proposition

Here as an alternative proof.

**Proof 2– Binomial Theorem.**

$$\begin{aligned} 2^n &= (1+1)^n \\ &= \sum_{k=0}^{n} (1)^k (1)^{n-k} \binom{n}{k} \\ &= \sum_{k=0}^{n} \binom{n}{k}. \end{aligned}$$





## Proposition

**Proposition.** (Theoretical Exercise 13)

Show that for $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

**Solution.** Use the binomial theorem:

$$\begin{aligned} 0 &= (-1+1)^n \\ &= \sum_{i=0}^{n} (-1)^i (1)^{n-i} \binom{n}{i} \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i}. \end{aligned}$$

# Problem

**Problem.** A student has 12 books and may keep 6, sell 4 and donate 2 to the library. How many outcomes are there

**Method 1.** Break the problem into three steps:

1. There are $\binom{12}{6}$ ways of choosing 6 books to keep.
2. There are $\binom{6}{4}$ ways of choosing 4 out of remaining 6 to sell.
3. There is 1 way remaining to choose 2 books to give away.

The number of outcomes is $\binom{12}{6} \cdot \binom{6}{4} = 13,860$.

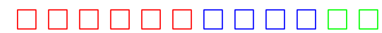




# Problem: Alternative method

**Problem.** A student has 12 books and may keep 6, sell 4 and donate 2 to the library. How many outcomes are there

**Method 2.** You want to sort 12 books into three colors:



where the colors mean keep, sell and give away.

There are 12! arrangements, but this over counts by

- 6! ways of arranging the books to keep,
- 4! ways of arranging the books to sell,
- 2! ways of arranging the books to give away.

The total number of outcomes is

$$\frac{12!}{6! \cdot 4! \cdot 2!} = 13,860.$$





# Principle

**Principle**

*Suppose $n$ objects must divided into $k$ groups of size $n_1, n_2, \ldots, n_k$, respectively, where $n = n_1 + n_2 + \ldots + n_k$. The number of ways of dividing the objects into groups is*

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$





# Notation

The following generalizes the binomial coefficients.

**Notation**

*If $n_1 + n_2 + \ldots + n_k = n$, then we write*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

*$\binom{n}{n_1,n_2,\ldots,n_k}$ is the number of ways of dividing $n$ objects into groups of sizes $n_1, n_2, \ldots, n_k$, respectively.*

**Remark.** The numbers $\binom{n}{n_1,n_2,\ldots,n_k}$ are called multinomial coefficients.

(See the statement of the Multinomial Theorem on p. 12 in Ross.)

# Problem

**Problem.** You are loading your 10 CD player with music for a party. You want 2 CDs from each of the following genres:

5 Classical 4 Country 6 Hip-Hop
7 Blues 8 Jazz

How many outcomes (in order of play) are there?

**Solution.** The procedure is as follows

1. Choose an arrangement of the genres with 2 choices for each:

$$\binom{10}{2,2,2,2,2}$$

2. Choose the 2 CDs per genre (order matters): $5 \cdot 4$ per genre, or $(5 \cdot 4)^5$ for all five genres.

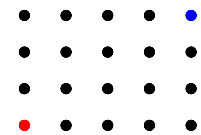
The total number of arrangements is $\binom{10}{2,2,2,2,2} \cdot (5 \cdot 4)^5$ .





# Problem

**Problem.** How many paths from • to • are possible, given that you can only go right one step or up one step on each move?



**Solution.** Each path requires 7 steps, divided into 3 steps up and 4 steps right (in some order). So, the number of paths is

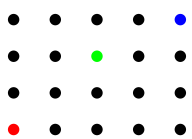
$$\binom{7}{3,4} = 35.$$





# Problem

**Problem.** How many paths from • to • that pass through • are possible, given that you can only go right one step or up one step on each move?



**Solution.** Break this into two problems:

**Step 1.** Paths from • to •: $\binom{4}{2,2}$.

**Step 1.** Paths from • to •: $\binom{3}{1,2}$.

The total number of paths is

$$\binom{4}{2,2} \cdot \binom{3}{1,2} = 18.$$





# Problem – Method 1

**Problem .** How many ways are there of choosing a committee of any size together with a chairperson from $n$ members?

**Method 1.** This is the global method. The procedure has two steps.

1. Choose a chairperson: $n$ outcomes.
2. Choose a committee from the remaining $n-1$: $2^{n-1}$ outcomes.

So, the total number of outcomes is

$$n2^{n-1}$$

# Problem – Method 2

**Method 2**. This is the global method. For each $k \geq 1$, solve the problem of the number of outcomes if the committee has $k$ members.

1. Choose a committee of size $k$: $\binom{n}{k}$ outcomes.
2. Choose a chairperson from those selected: $k$ outcomes.

There are $k \cdot \binom{n}{k}$ of possible subcommittees of size $k$.
By the Sum Rule, the number of outcomes is

$$\sum_{k=1}^{n} k\binom{n}{k}.$$

Since the two methods give the same value,

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}.$$