Math 425
Intro to Probability
Lecture 26

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Independent Random Variable

Recall. Two events E and F are independent if and only if

$$\mathbf{P}\{E \cap F\} = \mathbf{P}\{E\} \cdot \mathbf{P}\{F\}.$$

Equivalently,

$$\mathbf{P}(E \mid F) = \mathbf{P}\{E\}.$$

That is, knowing F does not change the probability of E.

Independent Random Variable

Definition

Let X and Y be random variables over the same sample space with joint distribution $F(x, y)$ and marginal distribution $F_X(x)$ and $F_Y(y)$.

X and Y are independent iff

$$F(a, b) = F_X(a) \cdot F_Y(b) \qquad \text{for all reals } a \text{ and } b.$$

Equivalently,

$$\mathbf{P}\{X \le a,\ Y \le b\} = \mathbf{P}\{X \le a\} \cdot \mathbf{P}\{Y \le b\} \qquad \text{for all reals } a \text{ and } b.$$

Key Property

☞ Checking random variables are independent requires checking infinitely many events are independent. We will shortly see a simple test for independence.

Independence extends from basic events to all events.

Theorem

If X and Y are independent random variables, then for any events $\{X \in A\}$ and $\{Y \in B\}$,

$$\mathbf{P}\{X \in A,\ Y \in B\} = \mathbf{P}\{X \in A\} \cdot \mathbf{P}\{Y \in B\}.$$

In particular, for all $-\infty \le a \le b \le \infty, -\infty \le c \le d \le \infty$,

$$\mathbf{P}\{a < X \le b,\ c < Y \le d\} = \mathbf{P}\{a < X \le b\} \cdot \mathbf{P}\{c < Y \le d\}$$

Proof

I will prove the theorem for intervals.
The first identity is from Lecture 25 (s. 12) or Ross, p.259.

$$\begin{aligned}
\mathbf{P}\{a < X \le b,\ c < Y \le d\} &= F(b,d) + F(a,c) - F(a,d) - F(b,c) \\
&= F_X(b)F_Y(d) + F_X(a)F_Y(c) \\
&\qquad - F_X(a)F_Y(d) - F_X(b)F_Y(c) \\
&= F_X(b)(F_Y(d) - F_Y(c)) \\
&\qquad - F_X(a)(F_Y(d) - F_Y(c)) \\
&= (F_X(b) - F_X(a)) \cdot (F_Y(d) - F_Y(c)) \\
&= \mathbf{P}\{a < X \le b\} \cdot \mathbf{P}\{c < Y \le d\}.
\end{aligned}$$

Equivalence: discrete case

☞ For discrete random variables, it is often easier to check the following equivalent condition for independence.

Theorem

Let X and Y be discrete random variables with joint probability mass function $p(x,y)$ and marginal probability mass functions $p_X(x)$ and $p_Y(y)$.

X and Y are independent if and only if

$$p(x,y) = p_X(x) \cdot p_Y(y).$$

Example: discrete case

Example. Let X and Y be discrete random variables with joint distribution

$$p(i,j) = \begin{cases} \frac{1}{36} & \text{if } 1 \le i,j \le 6 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the marginal mass functions

$$\begin{aligned}
p_X(i) &= \sum_{j=1}^{6} p(i,j) = \frac{1}{6} \qquad \text{if } 1 \le i \le 6 \\
p_Y(j) &= \sum_{i=1}^{6} p(i,j) = \frac{1}{6} \qquad \text{if } 1 \le j \le 6
\end{aligned}$$

and both are 0 otherwise.
☞ X and Y are independent since

$$p(i,j) = p_X(i) \cdot p_Y(j) \qquad \text{for all values } i \text{ and } j.$$

Equivalence: continuous case

☞ For continuous random variables, it is often easier to check the following equivalent condition for independence.

Theorem

Let X and Y be continuous random variables with joint density function $f(x,y)$ and marginal density functions $f_X(x)$ and $f_Y(y)$.

X and Y are independent if and only if

$$f(x,y) = f_X(x) \cdot f_Y(y).$$

Example: continuous case

Example. Let (X, Y) have a joint density given by

$$f(x,y) = \begin{cases} \lambda\mu e^{-\lambda x-\mu y}, & \text{if } 0 \le x, y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

☞ Compute the marginal densities $f_X(x)$ and $f_Y(y)$.

$$\begin{aligned} f_X(x) &= \int_0^\infty \lambda\mu e^{-\lambda x-\mu y}\, dy \\ &= \lambda e^{-\lambda x}\int_0^\infty \mu e^{-\mu y}\, dy = \lambda e^{-\lambda x} \\ f_Y(y) &= \int_0^\infty \lambda\mu e^{-\lambda x-\mu y}\, dx = \mu e^{-\mu y}. \end{aligned}$$

and both are 0 otherwise.

☞ X and Y are independent since

$$f(x,y) = f_X(x)\cdot f_Y(y) \qquad \text{for all values } x \text{ and } y.$$

General condition for Independence

☞ ☠ Warning. The following Proposition provides a more general condition for independence, however it is easily misapplied. I think it is best to compute the marginal density (mass) function, and verify independence using the other two theorem.

Proposition (Ross, Proposition 2.1)

The continuous (discrete) random variables X and Y are independent if and only if their joint probability density (mass) function can be expressed as

$$f_{X,Y}(x,y) = h(x)\cdot g(y) \qquad \textit{for all reals } x \textit{ and } y.$$

Note. The function $g(x)$ and $h(y)$ are NOT the marginal densities for X and Y, although the actual marginal densities will be

$$f_X(x) = C_1 h(x) \quad f_Y(y) = C_2 g(y) \qquad \text{for some constants } C_1 \text{ and } C_2.$$

Proof

☞ If X and Y are independent, then

$$f_{X,Y}(x,y) = f_Y(x)\cdot f_Y(y) \qquad \text{for all } x \text{ and } y.$$

☞ Conversely, suppose that

$$f_{X,Y}(x,y) = h(x)\cdot g(y) \qquad \text{for all } x \text{ and } y.$$

Then

$$\begin{aligned} 1 &= \int_{-\infty}^\infty\int_{-\infty}^\infty f_{X,Y}(x,y)\, dx\, dy \\ &= \int_{-\infty}^\infty h(x)\, dx \int_{-\infty}^\infty g(y)\, dy \\ &= C_1\cdot C_2 \end{aligned}$$

where $C_1 = \int_{-\infty}^\infty h(x)\, dx$ and $C_2 = \int_{-\infty}^\infty g(y)\, dy$.

Proof – continued

$$1 = C_1\cdot C_2 \qquad C_1 = \int_{-\infty}^\infty h(x)\, dx \qquad C_2 = \int_{-\infty}^\infty g(y)\, dy$$

☞ Compute the marginal distributions.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^\infty f_{X,Y}(x,y)\, dy = C_2 h(x) \\ f_Y(y) &= \int_{-\infty}^\infty f_{X,Y}(x,y)\, dx = C_1 g(y) \end{aligned}$$

So,

$$f_{X,Y}(x,y) = h(x)\cdot g(y) = \frac{f_X(x)}{C_2}\cdot\frac{f_Y(y)}{C_1} = f_X(x)\cdot f_Y(y).$$

Therefore, X and Y are independent.

Example

Example. Let (X, Y) have the joint distribution

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $h(x)$ and $g(y)$ be the functions

$$h(x) = -27 \qquad g(y) = -\frac{1}{27} \qquad \text{if } 0 < x, y < 1,$$

and are 0 otherwise. Then X and Y are independent since

$$f_{X,Y}(x,y) = h(x) \cdot g(y) \qquad \text{for all } x \text{ and } y.$$

But the marginal densities for X and Y are

$$f_X(x) = 1 \qquad f_Y(y) = 1 \qquad \text{if } 0 < x, y < 1,$$

and are 0 otherwise.

Example

Example. A little care is needed to apply the Proposition.
Let X count successes and Y count failures in n Bernoulli trials. So,

$$p(i,j) = \begin{cases} n! \frac{p^i}{i!} \frac{(1-p)^j}{j!} & \text{if } i + j = n \\ 0 & \text{otherwise .} \end{cases}$$

X and Y are NOT independent, since $p(i,j)$ is NOT a product of two functions $g(j)$ and $h(i)$ for EVERY i and j.
Compute the marginal distributions:

$$\begin{aligned} p_X(i) &= \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \\ p_Y(j) &= \frac{n!}{j!(n-j)!} p^{n-j} (1-p)^j \\ p(i,j) &\neq p_X(i) \cdot p_Y(j) \qquad \text{for any } i \text{ and } j. \end{aligned}$$

Compare to Ross, Example 2f.

Example

Example

A hospital averages λ births per day. The probability that a newborn is a boy is p and $(1-p)$ that it is a girl. It is reasonable to model the random variable X denoting the number births in a day as a Poisson random variable. (Why?)

What are the marginal mass functions for B and G, denoting the number of boys and girls born in a day? Are they independent?

Example – Continued

☞ We need to compute $\mathbf{P}\{B = n,\ G = m\}$ for integers $0 \leq n, m$ to obtain the marginal mass functions and assess independence.
Conditionalize on $\{X = n + m\}$ (note that $X = B + G$):

$$\begin{aligned} \mathbf{P}\{B = n,\ G = m\} &= \mathbf{P}(B = n,\ G = m \,|\, X = n + m) \cdot \mathbf{P}\{X = n + m\} \\ &\quad + \mathbf{P}(B = n,\ G = m \,|\, X \neq n + m) \cdot \mathbf{P}\{X \neq n + m\} \\ &= \mathbf{P}(B = n,\ G = m \,|\, X = n + m) \cdot \mathbf{P}\{X = n + m\} \\ &= \mathbf{P}(B = n,\ G = m \,|\, X = n + m) \cdot \left(e^{-\lambda} \frac{(\lambda)^{n+m}}{(n+m)!} \right) \end{aligned}$$

☞ Given that there were $n + m$ births, the number of boys and girls are binomially distributed, so

$$\mathbf{P}(B = n,\ G = m \,|\, X = n + m) = \binom{n+m}{n} p^n (1-p)^m$$

Example – Continued

☞ Substituting

$$\begin{aligned}
\mathbf{P}\{B=n,\ G=m\} &= \mathbf{P}(B=n,\ G=m \mid X=n+m)\cdot\mathbf{P}\{X=n+m\} \\
&= \binom{n+m}{n} p^n(1-p)^m \cdot \left(e^{-\lambda}\frac{(\lambda)^{n+m}}{(n+m)!}\right) \\
&= \frac{(n+m)!}{n!m!} p^n(1-p)^m \cdot \left(e^{-\lambda}\frac{(\lambda)^{n+m}}{(n+m)!}\right) \\
&= e^{-\lambda}\frac{(\lambda p)^n\cdot(\lambda(1-p))^m}{n!m!} \\
&= \left(e^{-\lambda p}\frac{(\lambda p)^n}{n!}\right)\cdot\left(e^{-\lambda(1-p)}\frac{(\lambda(1-p))^m}{m!}\right)
\end{aligned}$$

Example – Continued

$$\mathbf{P}\{B=n,\ G=m\} = \left(e^{-\lambda p}\frac{(\lambda p)^n}{n!}\right)\cdot\left(e^{-\lambda(1-p)}\frac{(\lambda(1-p))^m}{m!}\right)$$

☞ The marginal mass functions:

$$\begin{aligned}
\mathbf{P}\{B=n\} &= \left(e^{-\lambda p}\frac{(\lambda p)^n}{n!}\right)\sum_m\left(e^{-\lambda(1-p)}\frac{(\lambda(1-p))^m}{m!}\right) \\
&= e^{-\lambda p}\frac{(\lambda p)^n}{n!} \\
\mathbf{P}\{G=m\} &= e^{-\lambda(1-p)}\frac{(\lambda(1-p))^m}{m!}
\end{aligned}$$

☞ B and G are independent since

$$\mathbf{P}\{B=n,\ G=m\} = \mathbf{P}\{B=n\}\cdot\mathbf{P}\{G=m\}$$

are Poisson random variables with parameters λp and $\lambda(p-1)$ – the expected number of boys and girls born in a given day.

Example – continued

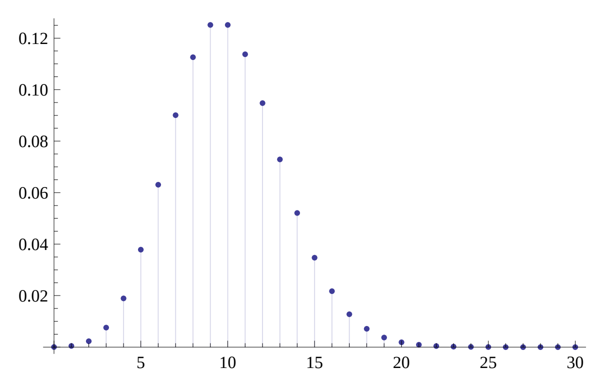
Example. Suppose the hospital averages $\lambda = 20$ births in a day, with boys and girls equally likely. On a given day there are 18 boys born. What is the probability that 10 girls were born on this day?

☞ Compute using the fact that B and G are independent, and Poisson distributed with parameter $\mu = 20\cdot\frac{1}{2} = 10$.

$$\begin{aligned}
\mathbf{P}(G=10 \mid B=18) &= \mathbf{P}\{G=10\} \\
&= e^{-10}\frac{10^{10}}{10!} \\
&\approx 0.1251
\end{aligned}$$

Example – continued

☞ The boys and girls born in a given day are Poisson distributed with parameter $\mu = 10$.



Example: broken stick

Example

We break a stick at random in two places, what is the probability that the three pieces form a triangle?

Assume that the stick has length 1, and that the breaks occur uniformly in the interval $(0, 1)$. Let the two break points be X and Y.

Example – continued

☞ (Euclid, I, Postulate 22) Three lengths can form a triangle if and only if the sum of any two are greater than the third.

☞ This rules out the following events

$$\{0 < X < Y \leq \frac{1}{2}\} \qquad \{\frac{1}{2} < X < Y \leq 1\}.$$

That is, X and Y must be on opposite side of the midpoint $\frac{1}{2}$

☞ We also need the following condition, so the third side is not too long,

$$|X - Y| < \frac{1}{2}.$$

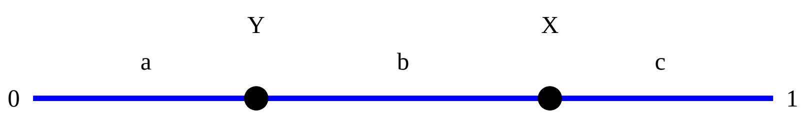
Example – continued

☞ If $a + b + c = 1$ and $a > b + c$, then $a > \frac{1}{2}$. But, our two conditions

$$(i) \qquad X < \frac{1}{2} < Y \quad \text{or} \quad Y < \frac{1}{2} < X$$

$$(ii) \qquad |X - Y| < \frac{1}{2}$$

guarantee this cannot happen:



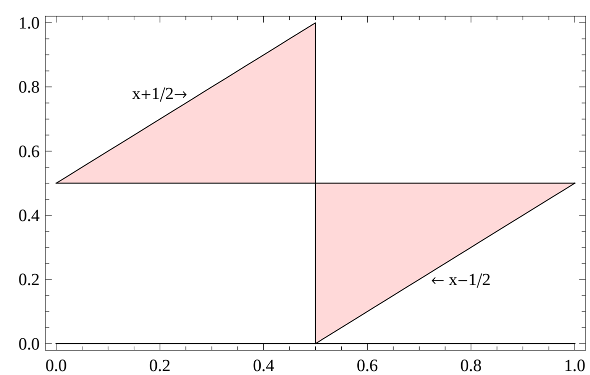
☞ $a, c < \frac{1}{2}$ by (i), and $b < \frac{1}{2}$ by (ii).

Example – continued

☞ The area of possible values of X and Y. Bounded between the lines

$$Y = X \pm \frac{1}{2} \qquad X = \frac{1}{2} = Y$$

The ratio of the area of this region to the area of the unit square $(= 1)$ is the probability.



Example – continued

☞ The area of each triangle is the same; so the area of the region is twice the area of the upper triangle:

$$2\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{x+\frac{1}{2}} dy\,dx = 2\int_0^{\frac{1}{2}} x\,dx = \frac{2}{8}.$$

The probability of breaking the stick in points X and Y so that it forms a triangle is $\frac{1}{4}$.

Joint Uniform Distribution

☞ Intuitively, we choose a point P “at random" from a region R if the probability the point lies in any subregion of R is proportional the area of the subregion.

Definition

A random variable (X, Y) is uniformly distributed over an integrable region R in the plane if its joint density is

$$f(x,y) = \begin{cases} (\text{area of } R)^{-1} & \text{if } (x,y) \in R \\ 0 & \text{otherwise .} \end{cases}$$

Example

Example. Let $P = (X, Y)$ be uniformly distributed in the rectangle $[0,1] \times [0,1]$. The joint density is

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise .} \end{cases}$$

The coordinate variables X and Y are independent, since $f(x, y)$ factors

$$f(x,y) = h(x) \cdot g(y)$$

where $h(x) = -27$ and $g(y) = -\frac{1}{27}$ when $0 \le x, y \le 1$ and 0 otherwise.

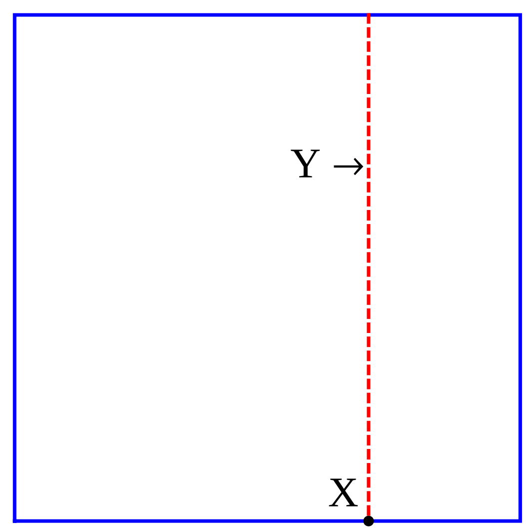
The actual marginal densities are

$$f_X(x) = \int_0^1 dy = 1$$

$$f_Y(y) = \int_0^1 dx = 1$$

Unit Square

☞ Coordinate variables (X, Y) are independent on the unit square.



Example

Example. Let $P = (X, Y)$ be uniformly distributed in the rectangle $[0, a] \times [0, b]$. The joint density is

$$f(x, y) = \begin{cases} \frac{1}{ab} & \text{if } 0 \le x \le a,\ 0 \le y \le b \\ 0 & \text{otherwise .} \end{cases}$$

The coordinate variables X and Y are independent.
The actual marginal densities are

$$f_X(x) = \int_0^b \frac{1}{ab}\, dy = \frac{1}{a}$$
$$f_Y(y) = \int_0^a \frac{1}{ab}\, dx = \frac{1}{b}$$

which is exactly what you would guess, since X is uniformly distributed in $[0, a]$ and Y in $[0, b]$.

Example

Example. Let $P = (X, Y)$ be uniformly distributed in the unit circle C. The joint density is

$$f(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } (x, y) \in C \\ 0 & \text{otherwise .} \end{cases}$$

However, X and Y are NOT independent, which you can verify from their marginal densities:

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2\sqrt{1-x^2}}{\pi}$$
$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx = \frac{2\sqrt{1-y^2}}{\pi}.$$

However, when $(x, y) \in C$

$$f(x, y) \neq f_X(x) \cdot f_Y(y).$$

Unit Circle

☞ Coordinate variables (X, Y) are independent on the unit square.