# Math 425
# Intro to Probability
# Lecture 24

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## Example: Roots

**Example**

Let $U$ be a uniformly distributed random variable on $[0,1]$. What is the probability that the equation

$$x^2 + 4Ux + 1 = 0$$

has two distinct real roots?

☞ The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are two real roots only when $b^2 - 4ac > 0$.





## Example: Roots

☞ We need to compute the values $U \in [0,1]$ such that

$$16U^2 - 4 > 0 \quad \text{equivalently} \quad U > \frac{1}{2}$$

Since $U$ is uniformly distributed on $[0,1]$, $\mathbf{P}\left\{U > \frac{1}{2}\right\} = \frac{1}{2}$.

☞ The probability that the equation

$$x^2 + 4Ux + 1 = 0$$

has two real roots when $U$ is chosen uniformly on $[0,1]$ is 0.5.





## Definition: Exponential density

**Definition**

A continuous random variable $T$ whose probability density function is given, for some parameter $\lambda > 0$, by

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } 0 \le t \\ 0 & \text{otherwise} \end{cases}$$

is said to be exponentially distributed with parameter $\lambda$.

The cumulative distribution function of the exponential density is given for $a > 0$ by

$$\begin{aligned} F_T(a) &= \mathbf{P}\{T \le a\} \\ &= \int_0^a \lambda e^{-\lambda t}\, dt \\ &= 1 - e^{-\lambda a}. \end{aligned}$$

## Survival Function

☞ Let $T$ be a continuous r.v. whose values are nonegative. The survival function is defined as

$$S(t) = \mathbf{P}\{T > t\}.$$

This is the probability that the item "survives" at least through time $t$.

☞ Let $\lambda(t)$ be the hazard rate (the likelihood of dying shortly after time $t$). Then

$$S(t) = \exp\left\{-\int_0^t \lambda(u)\,du\right\}.$$

☞ If $\lambda(t) = \lambda$ (a constant rate), then the survival function $S(t)$ determines an exponential distribution (the likelihood of living beyond time $t$).





## Example

**Example**

A 40-year-old woman nonsmoker can expect to live another 30 years; a 40-year-old woman smoker can expect to live half that time.

What is the probability that each lives another 10 years? 20 years? 30 years?

☞ Take the hazard rates to be constant: $\lambda_{ns}(t) = \lambda_{ns}$. So, the hazard rate for a smoker is $\lambda_s(t) = \lambda_s = \frac{1}{2}\lambda_{ns}$.

From the problem:

$$\lambda_{ns} = \frac{1}{30} \qquad \lambda_s = \frac{1}{15}$$





## Example – continued

☞ The survival functions are then

$$\begin{aligned} S_{ns}(t) &= e^{-\lambda_{ns} t} = e^{-\frac{1}{30}t} \\ S_s(t) &= e^{-\lambda_s t} = e^{-\frac{1}{15}t} \end{aligned}$$

☞ The probabilities of surviving another $t$ years are

| years | non-smoker | smoker |
|---|---|---|
| 10 | 0.717 | 0.513 |
| 20 | 0.513 | 0.264 |
| 30 | 0.368 | 0.135 |
| 40 | 0.264 | 0.069 |





## Example

**Example**

The lung cancer hazard rate of a $t$-year-old male smoker is given by

$$\lambda(t) = 0.027 + 0.00025(t-40)^2 \qquad t \geq 40.$$

What is the probability that a 40-year-old male survives to 50? 60? 70?

☞ The survival rate is given by

$$\begin{aligned} S(t) &= \exp\left\{-\int_{40}^t (0.027 + 0.00025(t-40)^2)\,dt\right\} \\ &= \exp\left\{-0.027(t-40) - 0.000083(t-40)^3\right\} \end{aligned}$$

# Example – continued

$$S(t) \quad = \quad \exp\left\{ -0.027(t-40) - 0.000083(t-40)^3 \right\}$$

☞ The probability of surviving until 50 is

$$S(50) \quad = \quad \exp\left\{ -0.027(10) - 0.000083(10)^3 \right\} \approx 0.7026.$$

☞ The probability of surviving until 60 is

$$S(60) \quad = \quad \exp\left\{ -0.027(20) - 0.000083(20)^3 \right\} \approx 0.3.$$

☞ The probability of surviving until 70 is

$$S(70) \quad = \quad \exp\left\{ -0.027(30) - 0.000083(30)^3 \right\} \approx 0.0473.$$





# Graph of Survival function

☞ Survival functions for 40-year-old cancer patients from previous examples. variable rate and constant rate with $\lambda = \frac{1}{15}$.

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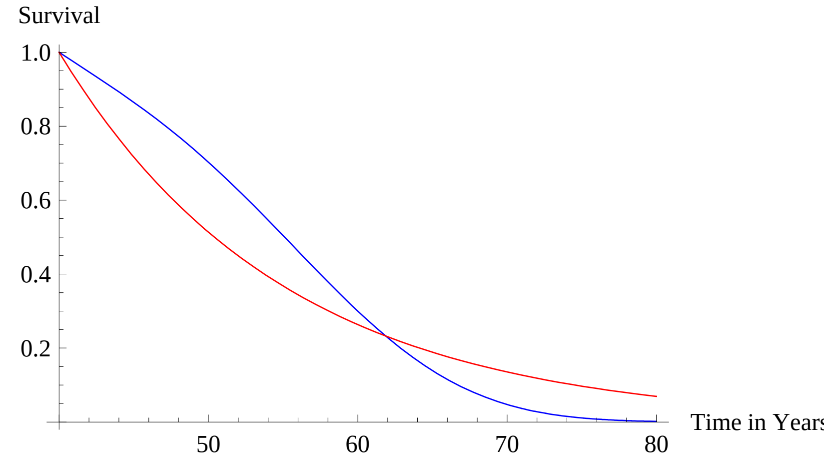







# Negative Binomial Distribution

☞ A negative binomial r.v. $X_k$ counts the number of trials in a Bernoulli trials process (where $p$ is the probability of success) until there are $k$ successes.

The probability mass function for $X$ is

$$\mathbf{P}\{X_k = n\} = \binom{n-1}{k-1} p^r (1-p)^{n-k} \qquad n \geq k$$

☞ The statistics are

$$E[X_k] \quad = \quad \frac{k}{p}$$

$$Var(X_k) \quad = \quad \frac{k(1-p)}{p^2}$$

**Note**. When $k = 1$, $X_1$ is a geometric random variable.





# Waiting time for Poisson process

☞ The continuous exponential random variable $T$ is waiting time for the first success in a Poisson process. (The continuous version of a Bernoulli process.)

We want the want the waiting time $T_k$ for $k$ successes in a Poisson process (the continuous version of the negative binomial r.v.).

☞ Let $N_t$ count the number of successes in a Poisson process that occur in the interval $[0, t]$. From Chapter 4 (§7) we saw $N_t$ is a Poisson random variable.

The key is that

$$T_k > t \quad \text{if and only if} \quad N_t < k$$

## Cumulative Distribution for Waiting Time

☞ Since $T_k > t$ exactly when $N_t < k$:

$$\begin{aligned} \mathbf{P}\{T_k > t\} &= \mathbf{P}\{N_t < k\} \\ &= \sum_{j=0}^{k-1} \mathbf{P}\{N_t = j\} \\ &= \sum_{j=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \end{aligned}$$

Equivalently, the cumulative distribution of $T_k$ is

$$\mathbf{P}\{T_k \le t\} = 1 - \sum_{j=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

**Note**. When $k = 1$, $T_1$ is an exponential random variable.





## Density for Gamma Distribution

☞ The cumulative distribution of $T_k$ (for $t \ge 0$) is

$$F_{T_k}(t) = \mathbf{P}\{T_k \le t\} = 1 - \sum_{j=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}.$$

The density function for $T_k$ is obtained by differentiating:

$$\begin{aligned} f_{T_k}(t) &= \sum_{j=0}^{k-1} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} - \sum_{j=1}^{k-1} e^{-\lambda t} \lambda j \frac{(\lambda t)^{j-1}}{j!} \\ &= \sum_{j=0}^{k-1} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} - \sum_{j=1}^{k-1} e^{-\lambda t} \lambda \frac{(\lambda t)^{j-1}}{(j-1)!} \\ &= \sum_{j=0}^{k-1} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} - \sum_{j=0}^{k-2} e^{-\lambda t} \lambda \frac{(\lambda t)^j}{j!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}. \end{aligned}$$





## Gamma distribution

☞ The random variable $T_k$ which counts the time required for $k$ events to occur in a Poisson process has cumulative distribution

$$F_{T_k}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - \sum_{j=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} & \text{if } t \ge 0; \end{cases}$$

and density function

$$f_{T_k}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} & \text{if } t \ge 0. \end{cases}$$

☞ Any random variable with this distribution is said to have a gamma distribution with parameters $(k, \lambda)$, where $\lambda > 0$ (the scale ) and $k$ is an nonegative integer (the shape).





## Gamma Density

☞ The density function for a gamma distributed r.v. with parameters $(k, \lambda)$ is a probability density function.

☞ The density is clearly nonnegative. We prove that it also takes the value 1 on the reals by induction on $k$. When $k = 1$, the density function is the same as the exponential density. Assume true for $k$.

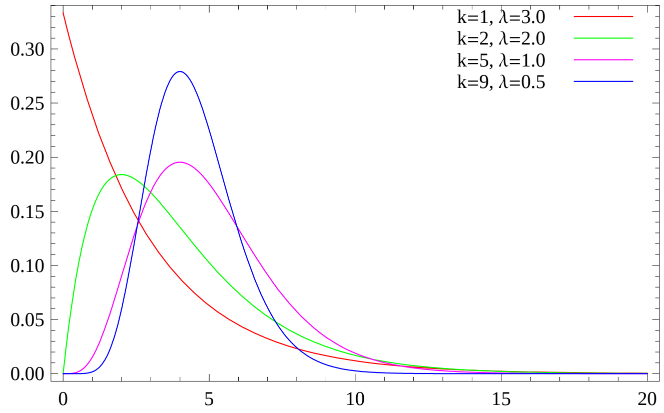
Use integration by parts with $u = \frac{(\lambda t)^k}{k!}$ and $dv = \lambda e^{-\lambda t}$.

$$\begin{aligned} \int_0^\infty \lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} &= \left[ -e^{-\lambda t} \frac{(\lambda t)^k}{k!} \Big|_0^\infty \right] + \int_0^\infty \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \\ &= 0 + \int_0^\infty \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \\ &= 1 \end{aligned}$$

The last line is by the induction hypothesis for $k$.

## Gamma density

☞ Plot of gamma density for $(k, \lambda)$.
Note that the "hump" in the density function occurs near $k\lambda$, which is the expectation.

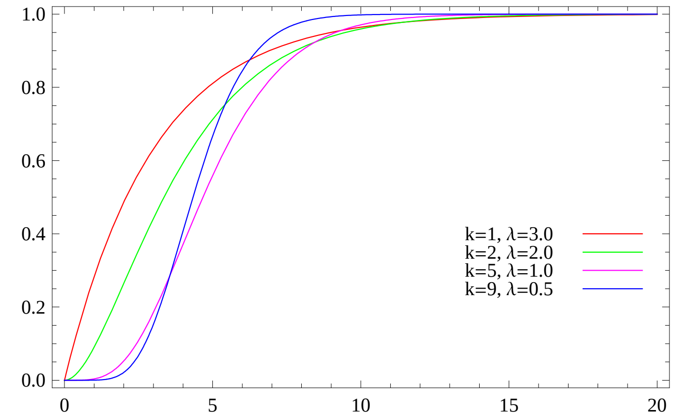







## Gamma distribution

☞ Plot of gamma cumulative distribution for $(k, \lambda)$.








## Example

**Example.** Customers arrive at a service station at a rate of 3 per hour. What is the probability that the second customer arrives after 1 hour?

**Solution.** Let $T$ be the time of arrival of the second customer. So, $T$ is gamma distributed with parameters $(k = 2, \lambda = 3)$. Let $t = 1$ in the gamma distribution:

$$\begin{aligned} \mathbf{P}\{T > 1\} &= e^{-3} + 3e^{-3} \\ &\approx 0.1991. \end{aligned}$$





## Example

**Example.** Defects in a type of wire follow the Poisson model, with rate 1 per 100 meter. Find the probability that the 5'th defect is located between 450 and 550 meters.

**Solution.** Let $L$ be the length of wire until the 5'th defect. So, $L$ is gamma distributed with parameters $(k = 5, \lambda = 1)$. In this problem, $t$ refers to length (in 100 meters)

$$\begin{aligned} \mathbf{P}\{4.5 < L < 5.5\} &= F_L(5.5) - F_L(4.5) \\ &= \left(1 - \sum_{j=0}^{4} e^{-5.5}\frac{(5.5)^j}{j!}\right) - \left(1 - \sum_{j=0}^{4} e^{-4.5}\frac{(4.5)^j}{j!}\right) \\ &= \sum_{j=0}^{4} e^{-4.5}\frac{(4.5)^j}{j!} - \sum_{j=0}^{4} e^{-5.5}\frac{(5.5)^j}{j!} \\ &= 0.1746 \end{aligned}$$

## Expectation

☞ Let $T$ be a gamma distribution with parameters $(k, \lambda)$.

$$\begin{aligned} E[T] &= \frac{1}{(k-1)!}\int_0^\infty (\lambda t)e^{-\lambda t}(\lambda t)^{k-1}\,dt \\ &= \frac{1}{(k-1)!}\int_0^\infty e^{-\lambda t}(\lambda t)^{k}\,dt \\ &= \frac{k!}{\lambda(k-1)!}\int_0^\infty \lambda e^{-\lambda t}\frac{(\lambda t)^k}{k!}\,dt \\ &= \frac{k}{\lambda} \end{aligned}$$





## Computing variance

☞ Let $T$ be a gamma distribution with parameters $(k, \lambda)$.

$$\begin{aligned} E[T^2] &= \frac{1}{(k-1)!}\int_0^\infty (\lambda t^2)e^{-\lambda t}(\lambda t)^{k-1}\,dt \\ &= \frac{1}{\lambda(k-1)!}\int_0^\infty (\lambda^2 t^2)e^{-\lambda t}(\lambda t)^{k-1}\,dt \\ &= \frac{1}{\lambda(k-1)!}\int_0^\infty e^{-\lambda t}(\lambda t)^{k+1}\,dt \\ &= \frac{(k+1)!}{\lambda^2(k-1)!}\int_0^\infty \lambda e^{-\lambda t}\frac{(\lambda t)^{k+1}}{(k+1)!}\,dt \\ &= \frac{k(k+1)}{\lambda^2} \end{aligned}$$





## Expectation and variance

☞ Let $T$ be a gamma distribution with parameters $(k, \lambda)$.

$$\begin{aligned} E[T] &= \frac{k}{\lambda} \\ E[T^2] &= \frac{k(k+1)}{\lambda^2} \\ Var(T) &= E[T^2] - \left(E[T]\right)^2 \\ &= \frac{k(k+1)}{\lambda^2} - \frac{k^2}{\lambda^2} \\ &= \frac{k}{\lambda^2} \\ SD(T) &= \sqrt{Var(T)} = \frac{\sqrt{k}}{\lambda}. \end{aligned}$$

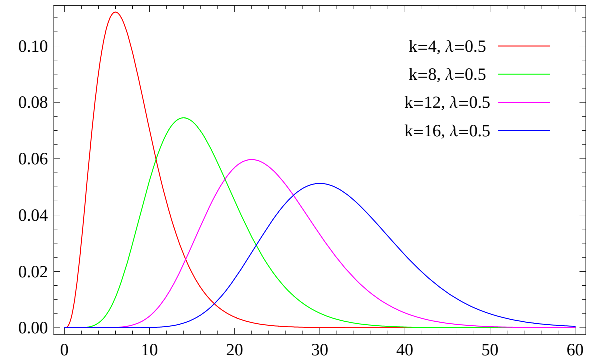




## Gamma density

☞ Plot of gamma density for $(k, \lambda = 0.5)$.

As $k$ increases “hump" gets closer to the mean $\frac{k}{\lambda}$, and the curve looks more “normal". By the Central Limit Theorem, the gamma distribution approaches the normal distribution as $k$ gets larger.



## Example

**Example.** Suppose that requests to a web server follow the Poisson model with rate $\lambda = 5$ . Compute the mean and standard deviation of the time of the 10th request.

**Solution.** Let $T$ be the waiting time until the 10th request. So, $T$ is gamma distributed with parameters $(k = 10, \lambda = 5)$.

$$E[T] = \frac{10}{5} = 2$$
$$Var(T) = \frac{10}{25}$$
$$SD(T) = \frac{\sqrt{10}}{5} \approx 0.6325.$$





## Example

**Example.** Suppose that Y has a gamma distribution with mean 40 and standard deviation 20. Find the shape parameter k and the rate parameter $\lambda$.

**Solution.**

$$E[Y] = 40 = \frac{k}{\lambda}$$
$$SD(Y) = 20 = \frac{\sqrt{k}}{\lambda}$$
$$Var(Y) = SD(Y)^2 = 400 = \frac{k}{\lambda^2}.$$

Solving:

$$\lambda = \frac{1}{10} \qquad k = 4.$$





## Definition of Gamma Function

☞ The gamma function (represented by the capitalized Greek letter Γ) is an extension of the factorial function to all positive real numbers.

**Definition**

For all real numbers $0 < \alpha$,

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy.$$

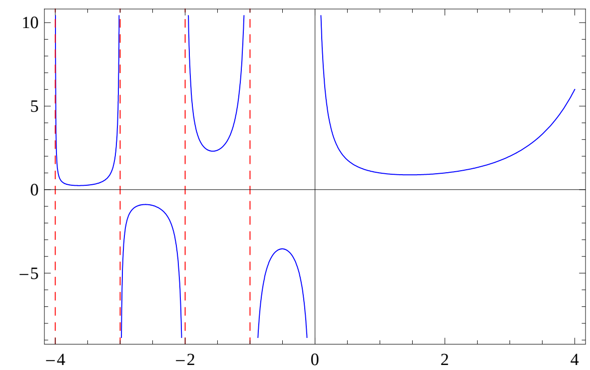
**Note.** $0 \le \Gamma(\alpha) < \infty$ for all $\alpha > 0$.





## Gamma Function Plot

☞ Plot of gamma function. There are asymptotes at all integers $z \le 0$.



## Properties of Gamma Function

☞ Some of the key properties of the gamma function. Note how similar these properties are to the factorial function. The most well-known value of the gamma function at non-integer arguments is given in (b).

**Theorem**

*For all real numbers* $0 < \alpha$*,*

(a) $\Gamma(1) = 1$.

(b) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(c) $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ *for all* $0 < \alpha$.

(d) $\Gamma(n) = (n-1)!$ *for all positive integers* $n = 1, 2, 3, \ldots$.





## Gamma Function

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy.$$

(a). For $\alpha = 1$,

$$\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1$$

(b). For $\alpha = \frac{1}{2}$, let $y = x^2$, $dy = 2x\, dx$

$$\begin{aligned}\Gamma(\alpha) &= 2\int_0^\infty e^{-x^2} x^{2\alpha-1}\, dx \\ \Gamma(\frac{1}{2}) &= 2\int_0^\infty e^{-x^2}\, dx \\ &= \int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}\end{aligned}$$





## Integral values of Gamma function

(c). Integration by parts: $u = y^{\alpha-1}$, $dv = e^{-y}$:

$$\begin{aligned}\Gamma(\alpha) &= \Big[ -e^{-y} y^{\alpha-1}\Big|_0^\infty + \int_0^\infty e^{-y}(\alpha-1) y^{\alpha-2}\, dy \Big] \\ &= (\alpha-1)\int_0^\infty e^{-y} y^{\alpha-2}\, dy \\ &= (\alpha-1)\Gamma(\alpha-1).\end{aligned}$$

(d). For integral values $\alpha = n$:

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\Gamma(n-2) \\ &= \cdots \\ &= (n-1)(n-2)\cdots 2 \cdot \Gamma(1) \\ &= (n-1)!\end{aligned}$$





## Computations of Gamma function

☞ Properties (b) and (c) allow closed form computation for $\Gamma(\frac{n}{2})$ for odd positive integers $2n+1$ (where $n \geq 0$):

$$\begin{aligned}\Gamma(\frac{1}{2}) &= \sqrt{\pi} \\ \Gamma(\frac{3}{2}) &= 2^{-1} \cdot \sqrt{\pi} \\ \Gamma(\frac{5}{2}) &= 2^{-2} \cdot 3 \cdot \sqrt{\pi} \\ \Gamma(\frac{7}{2}) &= 2^{-3} \cdot 5 \cdot 3 \cdot \sqrt{\pi} \\ \Gamma(\frac{2n+1}{2}) &= 2^{-n} \cdot (2n-1) \cdot (2n-3) \cdots 5 \cdot 3 \cdot \sqrt{\pi}\end{aligned}$$

# General Gamma distribution

**Definition**

A random variable has the gamma distribution with parameters $(\alpha, \lambda)$, where $\lambda > 0$ (the scale ) and $\alpha > 0$ (the shape), if its density function is given by

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{\lambda e^{-\lambda t}(\lambda t)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } t \geq 0. \end{cases}$$

☞ When $\alpha = k$ is an integer, then the gamma distribution is also known as the $k$-Erlang distribution – the waiting time for $k$ successes in a Poisson Process.

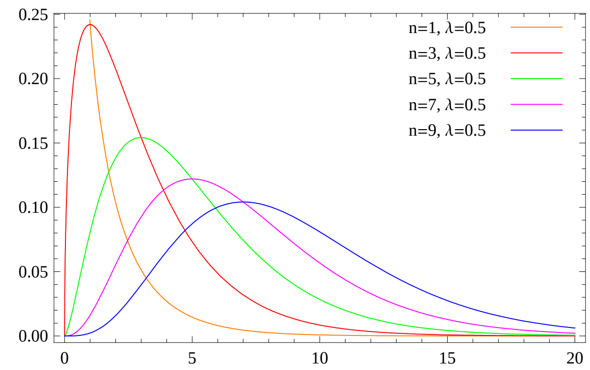
☞ When $\alpha = \frac{n}{2}$ and $\lambda = \frac{1}{2}$, the gamma distribution is known as the chi-squared distribution with $n$ degrees of freedom $(\chi^2_n)$. It is used to measure the goodness of fit of a known distribution to a theoretical one, and also to test for the independence of two criteria of classification.





# Chi-Squared Plot

☞ Plot of chi-squared distribution with $n = 2\alpha$ degrees of freedom (and $\lambda = 0.5$ are the parameters for the gamma distribution.








# Expectation and variance

☞ Let $X$ be a Gamma distribution with parameters $(\alpha, \lambda)$.

$$\begin{aligned} E[X] &= \frac{\alpha}{\lambda} \\ Var(X) &= \frac{\alpha}{\lambda^2} \\ SD(X) &= \sqrt{Var(X)} = \frac{\sqrt{\alpha}}{\lambda}. \end{aligned}$$

**Note**. The argument is exactly the same as for integer $\alpha$, using the fact (c):

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$