Math 425 Intro to Probability Lecture 23

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

March 11, 2009

Poisson Process: informal

☞ A Poisson process is an experiment in which events randomly occur continuously in time and independently of one another. The average number of events in a unit of time is λ, and it is assumed to be constant over the time of the experiment.

Examples. Some examples which are approximately Poisson processes

- radioactive decay,
- telephone calls arriving at a switchboard,
- page requests to a website,
- earthquakes in a region,
- meteor strikes.

Poisson Process: formal

☞ A Poisson process with parameter $\lambda > 0$ is a continuous time experiment that possesses the following properties

1. λ is the average number of occurrences of some event in a unit time interval. (λ is also called the intensity of the process.)
2. Independence: the number of occurrences counted in disjoint time intervals are independent of each other.
3. Stationary: the expected number of occurrences in an interval of time of length h depends only on the length h: it is $\lambda \cdot h$.
4. No counted occurrences are simultaneous.

☞ Compare the Poisson process to Lecture 17 (meteor showers) and Ross, p. 170. In Chapter 4 we fixed the length of time t allowed for the poisson process, and introduced a discrete Poisson random variable to count the number of occurrences in the time interval $[0, t]$.

Poisson Process and Poisson r.v.

☞ A Poisson process is the continuous-time counterpart to the discrete-time Bernoulli process.

☞ For each $t \geq 0$, let N_t be the Poisson random variable counting the number of events occurring in the interval of time $[0, t]$, where the average number of occurrences in a unit of time is λ.

N_t is a discrete r.v. (with parameter λ) having mass function

$$p_t(k) = \mathbf{P}\{N_t = k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

Note. This was proven in Ross p. 170, and argued for in Lecture 17.

Waiting Time: Cumulative Distribution

☞ How long must we wait for the first event in a Poisson process?
Let T be the random variable which counts the time we must wait (starting at $t = 0$) for the first event.

☞ If the first event occurs after time t, then there must have been no events in the interval $[0, t]$:

$$\Pr\{T > t\} = \mathbf{P}\{N_t = 0\} = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}.$$

☞ Equivalently, if the first occurs before time t, then there must have been at least one event in the interval $[0, t]$:

$$\Pr\{T \leq t\} = \mathbf{P}\{N_t \geq 1\} = 1 - \mathbf{P}\{N_t = 0\} = 1 - e^{-\lambda t}.$$

Waiting time: Density

☞ The waiting time T for the first event in a Poisson process with parameter λ, is a random variable with cumulative distribution

$$F_T(t) = \mathbf{P}\{T \leq t\} = 1 - e^{-\lambda t}$$

The density for T is

$$\begin{aligned} f_T(t) &= \frac{d}{dt}F_T(t) \\ &= \frac{d}{dt}(1 - e^{-\lambda t}) \\ &= \lambda e^{-\lambda t} \end{aligned}$$

Example

Example. A switchboard operator averages 10 calls an hour. What is the probability that the operator can take a 6 minute doughnut break?

Solution. This is a Poisson process with $\lambda = 10$.
Let T be the waiting time for the next call.

$$\mathbf{P}\{T > \frac{1}{10}\} = e^{-10\cdot\frac{1}{10}} = e^{-1} \approx 0.368.$$

There is a 36.8% chance of an uninterrupted doughnut break.

Definition: Exponential density

Definition

A continuous random variable T whose probability density function is given, for some parameter $\lambda > 0$, by

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } 0 \leq t \\ 0 & \text{otherwise} \end{cases}$$

is said to be an exponential random variable (or exponentially distributed) with parameter λ.

The cumulative distribution function of the exponential density is given for $a > 0$ by

$$\begin{aligned} F_T(a) &= \mathbf{P}\{T \leq a\} \\ &= \int_0^a \lambda e^{-\lambda t}\,dt \\ &= 1 - e^{-\lambda a}. \end{aligned}$$

Exponential density function

☞ Let $\lambda > 0$ in the exponential density:

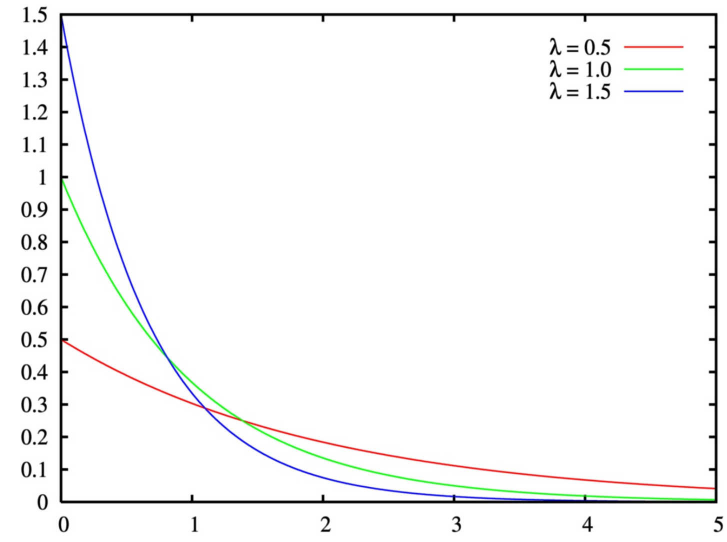
$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } 0 \leq t \\ 0 & \text{otherwise} \end{cases}$$

☞ The exponential density function is a probability density function.

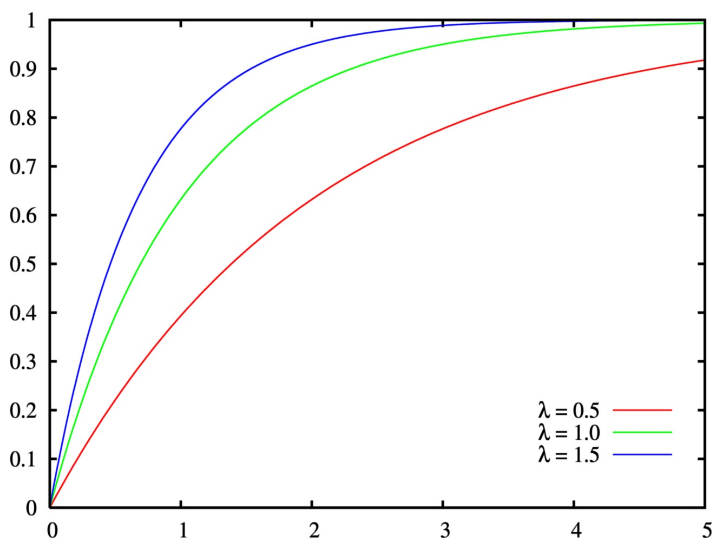
$$(a) \qquad f(t) \geq 0 \qquad \text{for all } t$$

$$\begin{aligned}(b) \qquad \int_{-\infty}^{\infty} f(t)\,dt &= \int_{0}^{\infty} \lambda e^{-\lambda t}\,dt \\ &= \lim_{M\to\infty} \left[-e^{-\lambda t}\Big|_{t=0}^{M} \right] \\ &= \lim_{M\to\infty} 1 - e^{-\lambda M} = 1\end{aligned}$$

Exponential Density



Exponential Distribution



Computation of Expectation

☞ Let T be an exponential random variable with parameter λ. Use integration by parts with $u = t$ and $dv = \lambda e^{-\lambda t}$.

$$\begin{aligned} E[T] &= \int_0^{\infty} \lambda t e^{-\lambda t}\,dt \\ &= \left[-te^{-\lambda t}\Big|_0^{\infty} \right] + \int_0^{\infty} e^{-\lambda t}\,dt \\ &= \left[\lim_{M\to\infty} -\frac{M}{e^{\lambda M}} \right] + \frac{1}{\lambda}\int_0^{\infty} \lambda e^{-\lambda t}\,dt \\ &= 0 + \frac{1}{\lambda} = \frac{1}{\lambda} \end{aligned}$$

since

$$\lim_{M\to\infty} -\frac{M}{e^{\lambda M}} = 0.$$

Computation of Variance

☞ Let T be an exponential random variable with parameter λ.
Use integration by parts with $u = t^2$ and $dv = \lambda e^{-\lambda t}$.

$$\begin{aligned} E[T^2] &= \int_0^\infty t^2 \lambda e^{-\lambda t}\, dt \\ &= \left[-t^2 e^{-\lambda t} \Big|_0^\infty \right] + \int_0^\infty 2te^{-\lambda t}\, dt \\ &= \left[\lim_{M\to\infty} -\frac{M^2}{e^{\lambda M}} \right] + \frac{2}{\lambda} \int_0^\infty \lambda t e^{-\lambda t}\, dt \\ &= 0 + \frac{2}{\lambda^2} = \frac{2}{\lambda^2} \end{aligned}$$

since

$$\lim_{M\to\infty} -\frac{M^2}{e^{\lambda M}} = 0.$$

Expectation and Variance

☞ The expectation and variance for an exponentially distributed r.v. T with paramter λ

$$\begin{aligned} E[T] &= \frac{1}{\lambda} \\ E[T^2] &= \frac{2}{\lambda^2} \\ Var(T) &= E[T^2] - \left(E[T]\right)^2 \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \\ SD(T) &= \sqrt{Var(T)} = \frac{1}{\lambda} \end{aligned}$$

Waiting Time and Exponential R.V.

☞ Let T be the waiting time for the first event of a Poisson process with an average number of λ events per unit time.
T is an exponentially distributed random variable.

☞ The expected waiting time is $E[T] = \frac{1}{\lambda}$.

Example. If our telephone operator averages 10 calls per hour, then the expected time to wait between calls is $\frac{1}{10}$ hours, or 6 minutes.

Memorylessness

☞ Consider a Bernoulli trial consisting of throwing a fair die. Let X be the geometric random variable counting the first time a "1" appears. The probability that the first "1" will appear in 4 more throws is independent of whether this is the first time you are throwing the die or the tenth time.

Definition

A discrete random variable X whose values lie in the set $\{0, 1, 2, \ldots\}$ is said to be memoryless if for any n and m

$$\mathbf{P}(X > m + n \,|\, X > m) = \mathbf{P}\{X > n\}.$$

A continuous random variable T is said to be memoryless if for any positive real numbers t and s

$$\mathbf{P}(T > s + t \,|\, T > s) = \mathbf{P}\{T > t\}.$$

Geometric R.V. is memoryless

☞ Let X be a geometrically distributed r.v. with probability p of success and q of failure.
X is memoryless.

☞ Since

$$\mathbf{P}\{X > k\} = \sum_{j=k}^{\infty} q^j p = q^k \sum_{j=0}^{\infty} q^j p = q^k,$$

it follows that

$$\mathbf{P}(X > m+n \mid X > m) = \frac{\mathbf{P}\{X > m+n\}}{\mathbf{P}\{X > m\}} = \frac{q^{m+n}}{q^m} = q^n.$$

Exponential R.V. is memoryless

☞ Let T be an exponentially distributed r.v. with parameter λ.
T is memoryless.

☞ Since

$$\mathbf{P}\{T > r\} = e^{-\lambda r}$$

it follows that

$$\mathbf{P}(T > s+t \mid T > s) = \frac{\mathbf{P}\{T > s+t\}}{\mathbf{P}\{T > s\}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}.$$

Unique memoryless distributions

☞ The geometric distribution is the only discrete probability distribution which is memoryless.

☞ The exponential distribution is the only continuous probability distribution which is memoryless.

See Ross, bottom of page 232 and footnote.

Example

Example. A busy repair shop can turn cars at 2 repairs per hour.
Let T be the exponentially distributed r.v. with parameter $\lambda = 2$.

☞ What is the probability that your repair with exceed half an hour?

$$\mathbf{P}\{T > \frac{1}{2}\} = e^{-\frac{1}{2}2} = e^{-1} \approx 0.368$$

☞ What is the probability that it exceeds 1 hour, given that the repair has taken half an hour so far?

$$\mathbf{P}(T > 1 \mid T > \frac{1}{2}) = \mathbf{P}\{T > \frac{1}{2}\} \approx 0.368.$$

Survival Function

☞ Let T be a continuous r.v. whose values are nonnegative, with cumulative distribution $F(t)$ and density $f(t)$. Take T to provide the time of death (or failure) of some item.

The survival function is defined as

$$S(t) = \mathbf{P}\{T > t\} = 1 - \mathbf{P}\{T \le t\} = 1 - F(t).$$

This is the probability that the item "survives" at least through time t.

☞ $S(t)$ is also called the reliability function, depending on whether you have in mind organisms (survival) or machines (reliable).

Basic Properties

$$S(t) = \mathbf{P}\{T > t\} = 1 - \mathbf{P}\{T \le t\}.$$

☞ Some properties of the survival function $S(t)$:

$$\begin{array}{lll} (a) & S(0) = 1 & \\ (b) & S(u) \le S(t) & \text{if } u > t \\ (c) & \lim_{t\to\infty} S(t) = 0. & \end{array}$$

Hazard Function

☞ The hazard rate (or failure rate) is defined to be

$$\lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{S(t)}$$

$\lambda(t)$ is the conditional probability of failing in the near future, given that it has survived until time t:

$$\begin{aligned} \mathbf{P}(t < T < t + dt \mid T \ge t) &= \frac{\mathbf{P}\{t < T < t + dt\}}{\mathbf{P}\{T > t\}} \\ &\approx \frac{f(t)\,dt}{1 - F(t)} \\ &= \lambda(t)\,dt. \end{aligned}$$

Survival function and hazard rate

☞ Given the hazard rate $\lambda(t)$, compute the survival function $S(t)$. By definition,

$$\lambda(t) = \frac{\frac{d}{dt}F(t)}{1 - F(t)}.$$

Integrating both sides

$$\log(1 - F(t)) = -\int_0^t \lambda(u)\,du + C$$

Raising to power e

$$1 - F(t) = e^C \exp\left\{-\int_0^t \lambda(u)\,du\right\}$$

Since $S(t) = 1 - F(t)$ and $S(0) = 1$, it follows that $C = 0$, so

$$S(t) = \exp\left\{-\int_0^t \lambda(u)\,du\right\}.$$

Example of failure rate

Example. The length of life in years, T, of a heavily used terminal in a student computer laboratory is exponentially distributed with failure rate $\lambda = 0.5$ years.

☞ The reliability function of T is then

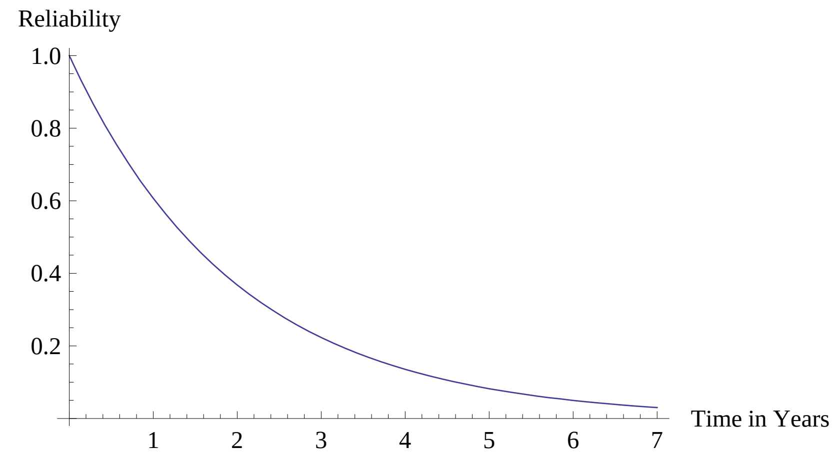
$$S(t) = \mathbf{P}\{T > t\} = \exp\left\{-\int_0^t 0.5\,du\right\} = e^{-0.5t}.$$

☞ The probability that a terminal will last more than 1 year is

$$S(1) = e^{-0.5} \approx 0.607$$

Graph of Reliability rate

☞ Reliability of terminal with failure rate $\lambda = 0.5$.



Hazard rate and Exponential function

Example. If the hazard function is constant, $\lambda(t) = \lambda$, this means the rate of failure does not depend on the time of survival. A t-year old item is just as likely to survive as a new item. What is the survival function?

$$\begin{aligned} S(t) &= \exp\left\{-\int_0^t \lambda(t)\,du\right\} \\ &= \exp\left\{-\int_0^t \lambda\,du\right\} \\ &= e^{-\lambda t} \end{aligned}$$

☞ Let T be the random variable giving the time of failure, and let $F(t) = 1 - S(t)$, so that $\mathbf{P}\{T \le t\} = F(t)$. But,

$$F(t) = 1 - e^{-\lambda t},$$

so T is exponentially distributed with parameter λ.

Constant rate failure

☞ In general, if the lifetime of a machine (or organism) is modeled by an exponential distribution with density

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

then λ is the failure rate of the machine, and the reliability at time t is

$$S(t) = e^{-\lambda t}.$$

☞ The expected life time of the machine is $\frac{1}{\lambda}$.
Because the exponential distribution is memoryless, we can think of breakdown as due to some random failure, not wear and tear.

Example

Example. Studies of a single-machine-tool system show that the time the machine operates before breaking down is exponentially distributed with a mean of 10 hours.

☞ Let T be the time until break down. Since $E[T] = 10$, the failure rate is $\lambda = 0.1$. The reliability function is $S(t) = e^{-0.1t}$.

☞ The probability the machine operates at least 12 hours is

$$S(12) = e^{-1.2} \approx 0.3011$$

☞ The probability the machine operates at least 12 hours given that it has lasted 8 is

$$\mathbf{P}(T > 12 \mid T > 8) = \mathbf{P}\{T > 4\} = e^{-0.4} \approx 0.6032.$$

Example

Example

A 40-year-old woman nonsmoker can expect to live another 30 years; a 40-year-old woman smoker can expect to live half that time.

What is the probability that each lives another 10 years? 20 years? 30 years?

☞ Take the hazard rates to be constant: $\lambda_{ns}(t) = \lambda_{ns}$. So, the hazard rate for a smoker is $\lambda_s(t) = \lambda_s = \frac{1}{2}\lambda_{ns}$.

From the problem:

$$\lambda_{ns} = \frac{1}{30} \qquad \lambda_s = \frac{1}{15}$$

Example – continued

☞ The survival functions are then

$$\begin{aligned} S_{ns}(t) &= e^{-\lambda_{ns} t} = e^{-\frac{1}{30}t} \\ S_s(t) &= e^{-\lambda_s t} = e^{-\frac{1}{15}t} \end{aligned}$$

☞ The probabilities of surviving another t years are

years	non-smoker	smoker
10	0.717	0.513
20	0.513	0.264
30	0.368	0.135
40	0.264	0.069

Example

Example

The lung cancer hazard rate of a t-year-old male smoker is given by

$$\lambda(t) = 0.027 + 0.00025(t - 40)^2 \qquad t \geq 40.$$

What is the probability that a 40-year-old male survives to 50? 60? 70?

☞ The survival rate is given by

$$\begin{aligned} S(t) &= \exp\left\{ -\int_{40}^{t} (0.027 + 0.00025(t-40)^2)\, dt \right\} \\ &= \exp\left\{ -0.027(t-40) - 0.000083(t-40)^3 \right\} \end{aligned}$$

Example – continued

$$S(t) = \exp\left\{ -0.027(t-40) - 0.000083(t-40)^3 \right\}$$

- The probability of surviving until 50 is

$$S(50) = \exp\left\{ -0.027(10) - 0.000083(10)^3 \right\} \approx 0.7026.$$

- The probability of surviving until 60 is

$$S(60) = \exp\left\{ -0.027(20) - 0.000083(20)^3 \right\} \approx 0.3.$$

- The probability of surviving until 70 is

$$S(70) = \exp\left\{ -0.027(30) - 0.000083(30)^3 \right\} \approx 0.0473.$$

Graph of Survival function

- Survival functions for 40-year-old cancer patients from previous examples. variable rate and constant rate with $\lambda = \frac{1}{15}$.

.