Math 425
Introduction to Probability
Lecture 2

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Review

Problem. A coach has ten players and wants to make two teams of five. How many ways are there of matching-up 5-on-5?

Method 1. Use the following procedure:

1. Choose 5 players out of 10 (order matters) : $\frac{10!}{(10-5)!}$ outcomes.
2. Order the remaining the remaining 5 players: $5!$ outcomes.

   So, there are $\frac{10!}{5!} \cdot 5! = 10! = 3,628,000$ outcomes

Method 2. Arrange the ten players in order and take the first five as one team and the second five as the other team.

There are 10! outcomes.

Review

Problem. The game *chuck-a-luck* involves throwing three dice. How many outcomes have three different values? Only two the same? All three the same?

Solution.

- There are 6 outcomes with all three values the same.
- There are $6 \cdot 5 \cdot 4 = 120$ values in which all have different values.
- Since there are $6^3 = 216$ total possible outcomes, there are 90 outcomes where only two dice agree.

Alternative. Let's directly compute the number of outcomes on when two dice agree. To distinguish the dice suppose they are colored red, green, and blue.

1. Choose 2 dice from 3 to agree: $\{r, g\}$, $\{r, b\}$, $\{g, b\}$.
2. There are 6 choices for the first dice of the pair, 1 choice for the second dice, and 5 choices for the third dice. So, $6 \cdot 5 = 30$ outcomes per pair.

Thus, there are $3 \cdot 30 = 90$ ways for only two dice to agree.

Problem

Problem. How many different arrangements can be formed from the letters $A\ A\ A\ B\ C$?

Solution. Break the problem into two steps.

1. Treat each instance of A as different: how many arrangement are there of $A_1\ A_2\ A_3\ B\ C$? $5!$.
2. We have over counted. For example, the arrangement $A\ A\ A\ B\ C$ has been over counted by $3!$:

$$\begin{array}{ccc} A_1\ A_2\ A_3\ B\ C & A_1\ A_3\ A_2\ B\ C & A_2\ A_1\ A_3\ B\ C \\ A_2\ A_3\ A_1\ B\ C & A_3\ A_1\ A_2\ B\ C & A_3\ A_2\ A_1\ B\ C \end{array}$$

There are $\frac{5!}{3!} = 20$ possible arrangements.

Problem

Problem. How many different letter arrangements can be formed from the letters $A\ A\ A\ B\ B$?

Solution. Break the problem into two steps.

1. Treat each instance of A and each instance of B as different: how many arrangement are there of $A_1\ A_2\ A_3\ B_1\ B_2$? $5!$.
2. We have over counted. For example, the arrangement $A\ A\ A\ B\ B$ has been over counted by $3!2!$:

$A_1\ A_2\ A_3\ B_1\ B_2$	$A_1\ A_3\ A_2\ B_1\ B_2$	$A_2\ A_1\ A_3\ B_1\ B_2$
$A_2\ A_3\ A_1\ B_1\ B_2$	$A_3\ A_1\ A_2\ B_1\ B_2$	$A_3\ A_2\ A_1\ B_1\ B_2$
$A_1\ A_2\ A_3\ B_2\ B_1$	$A_1\ A_3\ A_2\ B_2\ B_1$	$A_2\ A_1\ A_3\ B_2\ B_1$
$A_2\ A_3\ A_1\ B_1\ B_2$	$A_3\ A_1\ A_2\ B_1\ B_2$	$A_3\ A_2\ A_1\ B_2\ B_1$

There are $\frac{5!}{3!2!} = 10$ possible arrangements.

Principle

We generalize the last two problems.

Principle

The number of arrangments of n objects of which n_1 are alike, n_2 are alike, ..., n_k are alike are

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Remark. The number of arrangements of n distinct objects is

$$\frac{n!}{1!1!\cdots 1!} = n!.$$

Problem

Problem. How many different letter arrangements can be formed from the letters: $A\ B\ R\ A\ C\ A\ D\ A\ B\ R\ A$?

Solution. There are 11 letters: 5 A's, 2 B's, 1 C, 1 D, 2 R's. So, the number of arrangments is

$$\frac{11!}{5!2!1!1!2!} = 83,160.$$

Problem

Problem. How many poker hands (5 cards) are there?
Solution. Break the problem into two steps.

1. Count the number of different poker hands that can be dealt by order of the deal:
$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{(52-5)!}$$
2. Given 5 cards, there are 5! ways of ordering them. Since we only care about the cards dealt, not the order of the cards dealt, we overcounted each hand in Step 1 by $5!$.

Divide Step 1 by Step 2. The total number of poker hands is:

$$\frac{52!}{(52-5)!5!} = 2,598,960.$$

Combinations

Definition

A combination is a selection of objects from a set where order does not matter.

Principle

The number of combinations of r objects chosen from a set of n objects is

$$\frac{n \cdot (n-1) \cdots (n-r+1)}{r!} = \frac{n!}{(n-r)!r!}.$$

Notation

Notation

When $r \leq n$, we define

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

where we read $\binom{n}{r}$ as "n choose r".

$\binom{n}{r}$ is the number of subsets of size r that can be chosen from a set of size n (where the order of selection does not matter).

Convention

☞ By convention we define $0! = 1$.

☞ So, we have $\binom{n}{0} = \binom{n}{n} = 1$.

$$\binom{n}{0} = \frac{n!}{(n-0)!0!} = 1$$

$$\binom{n}{n} = \frac{n!}{(n-n)!n!} = 1$$

Problem

Problem. You are going on vacation, and can fit only 10 CDs in your case. You can choose the following genres

- 5 Classical
- 4 Country
- 6 Hip-Hop
- 7 Blues
- 8 Jazz

How many outcomes are there when you take 2 from each genre?

Solution. Choose two from each genre, in turn.

$$\binom{5}{2} \text{ Classical } \binom{4}{2} \text{ Country } \binom{6}{2} \text{ Hip-Hop } \binom{7}{2} \text{ Jazz } \binom{8}{2} \text{ Blues}$$

The number of outcomes is $\binom{5}{2} \cdot \binom{4}{2} \cdot \binom{6}{2} \cdot \binom{7}{2} \cdot \binom{8}{2} = 529,200$.

Problem for Monday

Problem. You are going on vacation, and can fit only 10 CDs in your case. You can choose the following genres

- 5 Classical
- 4 Country
- 6 Hip-Hop
- 7 Blues
- 8 Jazz

How ways of arranging the CDs you take (in order), when you take 2 from each genre?

Problem for Monday lecture.

Problem – part I

Problem. I am having a party, but I only have space for 5 of my 8 friends. How many possible arrangements of guests friends do I have?

Solution. $\binom{8}{5} = 56$.

Problem – part II

Problem. I am having a party, but I only have space for 5 of my 8 friends. Suppose 2 are feuding and won't attend together. How many arrangements of guests are there now?

Solution. Break the problem into separate ones we can solve.

1. If I invite neither friend, there are $\binom{6}{5}$ outcomes.
2. If I invite one feuding friend, there are $\binom{6}{4}$ outcomes per friend.

Each Step leads to different outcomes. By the Sum Rule there are

$$\binom{6}{5} + \binom{6}{4} + \binom{6}{4} = 36$$

outcomes.

Problem – part III

Problem. I am having a party, but I only have space for 5 of my 8 friends. Suppose two of my friends will only come together. How many arrangements of guests are there now?

Solution. Break the problem into separate ones we can solve.

1. If I invite neither friend, there are $\binom{6}{5}$ outcomes.
2. If I invite both friends, there are $\binom{6}{3}$ outcomes.

Each Step leads to different outcomes. By the Sum Rule there are

$$\binom{6}{5} + \binom{6}{3} = 26$$

outcomes.

Problem

Problem. How many ways can a Senate committee of 7 Democrats and 6 Republicans choose a subcommittee of 3 Democrats (one of which is the subcommittee chair) and 2 Republicans?

Method 1. The procedure is as follows:

1. Choose the chair from the Democrats: 7.
2. Choose the remaining 2 Democratic members: $\binom{6}{2}$.
3. Choose the 2 Republican members: $\binom{6}{2}$.

The number of outcomes: $7 \cdot \binom{6}{2} \cdot \binom{6}{2} = 1575$.

Method 2. The procedure is as follows:

1. Choose the 3 Democrats: $\binom{7}{3}$.
2. Choose the chair among these: 3.
3. Choose the 2 Republican members: $\binom{6}{2}$.

The number of outcomes: $3 \cdot \binom{7}{3} \cdot \binom{6}{2} = 1575$.

Problem

Problem. A teacher must choose 4 students for the math team from a class of 12. How many possible ways are there for making a team?

Method 1. Choose 4 students on the team: $\binom{12}{4} = 495$.

Method 2. Choose 8 students who are off the team: $\binom{12}{8} = 495$.

So, $\binom{12}{4} = \binom{12}{8}$.

Combinatorial identity

Fact

When $r \leq n$,

$$\binom{n}{r} = \binom{n}{n-r}$$

Reason.
There are two ways of choosing a group of r objects out of n possible.

Method 1. Choose r objects for the group: $\binom{n}{r}$.
Method 2. Choose the $n - r$ objects left-out of the group: $\binom{n}{n-r}$.

Since either method gives the same answer: $\binom{n}{r} = \binom{n}{n-r}$.

Problem

Problem. A school district has 3 schools which need 3 teachers each, and 9 teachers to fill these spots. How many different ways are there for assigning teachers to schools?

Solution. Transform to a problem we have solved:

1. Put nine balls in a bag: 3 red, 3 green, and 3 blue.
2. Distribute the nine balls to the nine teachers.
3. Let the color of the ball received determine the school to send the teacher.

The number of ways of distributing the 9 balls: 3 red, 3 green, and 3 blue, is

$$\frac{9!}{3!3!3!} = 1680.$$

Principle

Principle

Suppose n objects must divided into k groups of size $n_1, n_2, \ldots, n_k$, respectively, where $n = n_1 + n_2 + \ldots + n_k$. The number of ways of dividing the objects into groups is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Notation

The following generalizes the binomial coefficients.

Notation

If $n_1 + n_2 + \ldots + n_k = n$, then we write

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

$\binom{n}{n_1, n_2, \ldots, n_k}$ is the number of ways of dividing n objects into groups of sizes $n_1, n_2, \ldots, n_k$, respectively.

Remark. The numbers $\binom{n}{n_1, n_2, \ldots, n_k}$ are called multinomial coefficients.
(See the statement of the Multinomial Theorem on p. 12 in Ross.)