# Math 425
# Introduction to Probability
# Lecture 17

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## Two equations for $e^x$

☞ We will need the following two ways of specifying $e^x$.

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$





## Poisson distribution

**Definition**

A random variable $X$ taking on one of the values $0, 1, 2, \ldots$ is said to be a Poisson random variable with parameter $\lambda$ (where $\lambda > 0$) if

$$\mathbf{P}\{X = k\} = e^{-\lambda}\frac{\lambda^k}{k!} \qquad \text{for } k = 0, 1, 2, \ldots$$





## Poisson distribution

☞ Suppose $X$ is a Poisson random variable.
Since the events $\{X = k\}$ are mutually exclusive,

$$\begin{aligned}\mathbf{P}\{X \geq 0\} &= \sum_{k=0}^{\infty}\mathbf{P}\{X = k\}\\ &= \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\\ &= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\\ &= e^{-\lambda}\cdot e^{\lambda} = 1.\end{aligned}$$

☞ So, $p_X$ defines a probability distribution for $X$. (That is, $X$ takes its possible values with probability 1.)

# Ursid meteor shower

**Example**

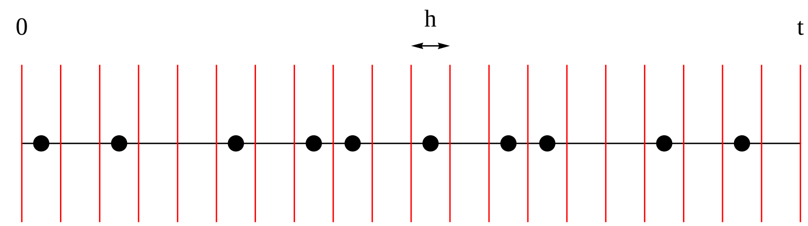
The Ursid meteor shower peaks around December 22 producing about 10 meteors per hour between 11pm and 4am. What is the probability that you will see a meteor if you start observing the night sky for fifteen minutes starting at midnight?





# Picture of the Poisson paradigm

☞ We are considering a time period $(0, t)$, $t = 1$ hour. Let $n$ be a VERY BIG number and subdivide the time period into $n$ VERY SMALL subintervals of length $h = \frac{t}{n}$ – small enough so that every interval contains at most one event (meteor strike).




☞ Now, we can think of the set-up as a Bernoulli trials process, where a trial is a subinterval and success is a meteor strike occurring in the subinterval of time.





# Poisson Paradigm

☞ Here is the general set-up, the Poisson Paradigm. We are supposed that the events are discrete, and that they occur in a medium (time, space, cookie dough, etc.) which is (approximately) indefinitely divisible.

(a) The average rate of occurrence in a measure of medium is $\lambda$.
(b) The probability of an event occurring in an interval is (approximately) proportional to the length of the interval
(c) The probability that an event occurring within an interval is independent of events that occur outside the interval.
(d) For sufficiently short intervals, the probability of more than one event occurring is very, very small.

☞ We will show that the random variable counting the number of events in the interval is an approximate Poisson random variable.





# Derivation – set-up

☞ Let $n$ be a large number, and subdivide the interval $t = 1$ into subintervals of equal length $h = \frac{t=1}{n}$. We choose $n$ so large that conditions (a-d) are true.

☞ Each subinterval is a trial. Success occurs in a trial if an event (meteor strike) occurs in the subinterval. This is a Bernoulli trials process:

1. Success (event occurs in the subinterval) and failure are mutually exclusive.
2. The probability of success is the same in each trial – by (a,b), this probability is $p = \lambda h = \lambda \frac{1}{n}$.
3. The trials are independent by (c).

☞ Let $X_n$ be the Binomial random variable which counts successes. By (d), the probability of two events occurring in the same subinterval is negligible, so we can take $X_n$ to be counting the number of events occurring in the interval $t = 1$.

## Derivation

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}\{X_n = k\} &= \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} \qquad p = \frac{\lambda}{n}\\ &= \lim_{n\to\infty} \left[\frac{n!}{(n-k)!k!}\right]\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}\\ &= \underbrace{\lim_{n\to\infty}\left[\frac{n!}{n^k(n-k)!}\right]}_{F}\left(\frac{\lambda^k}{k!}\right)\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\approx e^{-\lambda}}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\approx 1}\\ &= F\left(\frac{\lambda^k}{k!}\right)e^{-\lambda}\end{aligned}$$

Since $k$ is fixed,

$$F = \lim_{n\to\infty} \frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n} = 1$$

So,

$$\lim_{n\to\infty} \mathbf{P}\{X_n = k\} = e^{-\lambda}\frac{\lambda^k}{k!}$$





## Ursid meteor shower – derivation

☞ The meteor shower averages 10 per hour, and we will be observing for $t = \frac{1}{4}$ hour. The rate per 15 minute interval is

$$\lambda = 10 \cdot \frac{1}{4} = 2.5.$$

Let $X$ be the Poisson random variable with parameter $\lambda = 2.5$.

$$\begin{aligned}\mathbf{P}\{X \geq 1\} &= 1 - \mathbf{P}\{X < 1\} = 1 - \mathbf{P}\{X = 0\}\\ &= 1 - e^{-2.5} \approx 0.918\end{aligned}$$

There is a 92% chance I will see a meteor strike in my fifteen minutes of observation.





## Typical meteor showers

☞ A typical meteor shower has a peak of about 4 meteors per hour. So, the average for 15 minutes is $\lambda = 4 \cdot \frac{1}{4} = 1$.
Let $X$ be the Poisson random variable with parameter $\lambda = 1$.

$$\begin{aligned}\mathbf{P}\{X \geq 1\} &= 1 - \mathbf{P}\{X < 1\} = 1 - \mathbf{P}\{X = 0\}\\ &= 1 - e^{-1} \approx 0.63\end{aligned}$$

There is a 63% chance of observing a meteor in 15 minutes.





## Perseids meteor showers

☞ The Perseids shower has a peak of 60 meteors per hour. For a 15 minute interval, $\lambda = 60 \cdot \frac{1}{14} = 15$. The probability of seeing at least 7 meteors in 15 minutes is

$$\begin{aligned}\mathbf{P}\{X \geq 7\} &= 1 - \mathbf{P}\{X < 6\}\\ &= 1 - e^{-15}\left(1 + 15 + \frac{15^2}{2!} + \frac{15^3}{3!} + \frac{15^3}{3!} + \frac{15^4}{4!} + \frac{15^5}{5!} + \frac{15^6}{6!}\right)\\ &\approx 0.992\end{aligned}$$

## Law of Rare Events

☞ Rare events that occur independently but consistently in some region of time or space (or cookie dough) will often follow a Poisson distribution.

For this reason it is sometimes called the law of rare events.

- Meteors in the night sky,
- Bomb strikes on London during WWII,
- Soldiers killed by horse-kicks in the Prussian calvary,
- Accidents along a stretch of road,
- Currants in a scone (chips in a cookie),
- Calls to a switchboard,
- Dog biscuits purchased in a grocery store,
- Mutations in a given stretch of DNA,
- Misprints in a newspaper.





## Approximating a Binomial random variable

☞ We proved that fixing $k$ and letting $p = \frac{\lambda}{n}$
(where $\lambda$ is the average number of successes)

$$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = e^{-\lambda}\frac{\lambda^k}{k!}$$

☞ This means that we can use the Poisson random variable with parameter $\lambda$ to approximate a Binomial random variable when

$n$ is very large and $p$ is very small,
so that $\lambda = p \cdot n$ is of moderate size.

It is not necessary that the physical conditions (a-d) be true.

☞ The rule of thumb is that the Poisson distribution is a good approximation of the binomial distribution if $n \geq 20$ and $p \leq 0.05$.
So, $\lambda \leq 10$.





## Example: Typesetting

Example

Suppose that for a certain probability text book whose length is $n = 500$ pages, the probability of a misprint on a page is estimated at about $p = 0.002$.

What is the probability that fewer than two pages have misprints?





## Example: Typesetting

☞ The set-up is a Bernoulli trials process: each page is a trial, and the occurrence of at least one misprint on a page is a success. We also assume that the pages are independent.

☞ We could use a Binomial random variable and add up the probabilities of getting exactly zero and exactly one page of misprints:

$$(0.002)^0(0.998)^{500} + \binom{500}{1}(0.002)^1(0.998)^{499}$$

☞ It is much easier to use a Poisson random variable to approximate the desired Binomial probability.

The Poisson parameter is $\lambda = (0.002)(500) = 1$, so

$$e^{-1} + e^{-1}\frac{1}{1!} = 2e^{-1} \approx 0.73579$$

# Comparing Poisson and Binomial distributions

☞ In our example $n = 500$ is large and $p = 0.002$ is small, so that the average number of pages with misprints, $\lambda = (0.002)(500) = 1$ is moderate.

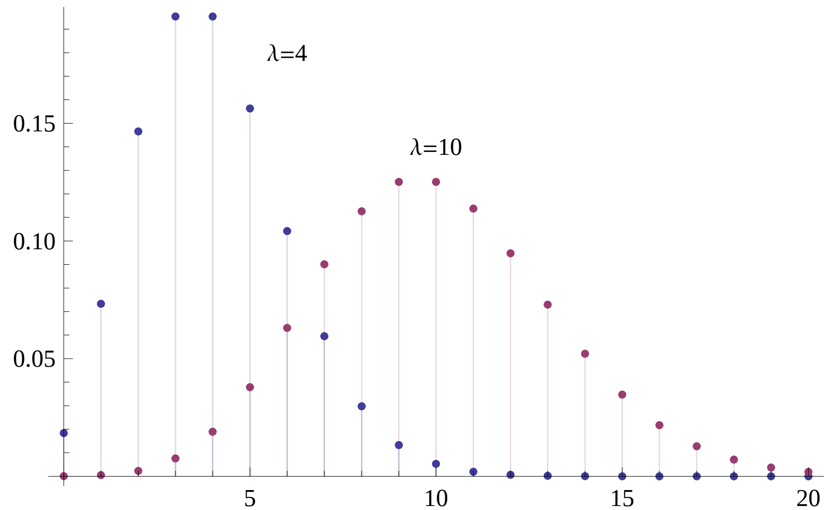
| $k$ | Poisson ($\lambda = 1$) | Binomial ($n = 500, p = 0.002$) |
|---|---|---|
| 0 | 0.36798 | 0.36751 |
| 1 | 0.36788 | 0.36825 |
| 2 | 0.18394 | 0.18412 |
| 3 | 0.06131 | 0.06125 |
| 4 | 0.01532 | 0.01525 |
| 5 | 0.00307 | 0.00303 |
| 6 | 0.00051 | 0.00050 |
| 7 | 0.00007 | 0.00007 |
| 8 | 0.00000 | 0.00000 |





# Poisson Distribution

☞ Poisson distributions for $\lambda = 4$ and $\lambda = 10$. Note how symmetric these are around $\lambda$. This is very similar to Binomial distributions around their mean.








# Mean and Variance

☞ Let $B$ be a binomial random variable with parameters $n$ and $p$. $B$ is approximated by a Poisson r.v. $X$ (parameter $\lambda = np$).

Since $E[B] = np$, we expect $E[X] = \lambda$.

**Theorem**

*Let $X$ be a Poisson random variable with parameter $\lambda$, where $0 < \lambda$.*

*The mean and variance of $X$ are given by*

$$E[X] = \lambda$$

$$Var(B) = \lambda$$





# Example

**Example**

The Boeing 747-400 has a seating capacity of 400 passengers. Acme Airways (AA) has overbooked a flight with 403 reservations. Any passenger independently fails to show up with probability 0.01.

What is the probability that AA will have to bump at least one passenger?

☞ Since $n$ is large and $p$ is small we can use a Poisson approximation with $\lambda = 403(0.01) = 4.03$ for the random variable $X$ counting passengers which don't show.

$$\begin{aligned} \mathbf{P}\{X < 3\} &= e^{-4.03}\left(1 + 4.03 + \frac{(4.03)^2}{2!}\right) \\ &\approx 0.234 \end{aligned}$$

So, there is a 23% chance some passenger will be bumped.

## Example

**Example**

Acme Cookie Company puts a bag of 600 chocolate chips into each batch of cookie dough. They thoroughly mix each batch, which makes 500 cookies.

What is the probability that a cookie is chipless?

☞ Since $n$ is large and $p$ is small we can use a Poisson approximation with $\lambda = 600(\frac{1}{500}) = 1.2$ for the random variable $X$ counting chips in each cookie.

$$\mathbf{P}\{X = 0\} = e^{-1.02} \approx 0.301.$$

There is a 30% chance that a chocolate chip cookie is sans chocolate chips.





## Example – continued

☞ What is the probability that a typical Acme cookie is chocolate saturated (at least 4 chips)?

$$\begin{aligned}\mathbf{P}\{X \geq 4\} &= 1 - \mathbf{P}\{X < 4\} \\ &= 1 - e^{-1.02}(1 + 1.2 + \frac{(1.2)^2}{2} + \frac{(1.2)^3}{3!}) \approx 0.034\end{aligned}$$

There is only a 3% chance it is chocolate saturated.

☞ Famous Alf's chocolate chip cookie averages $\lambda = 4$ chips per cookie.

$$\begin{aligned}\mathbf{P}\{X = 0\} &= e^{-4} \approx 0.0183 \\ \mathbf{P}\{X \geq 4\} &= 1 - e^{-4}(1 + 4 + \frac{(4)^2}{2} + \frac{(4)^3}{3!}) \approx 0.5665\end{aligned}$$





## Example

**Example**

In one of the earliest studies of the Poisson distribution, von Bortkiewicz (1898) considered deaths from mule kicks in the Prussian army corps. He collected data from 14 corps over a 20-year period.

| # Deaths | # corps with $x$ deaths in a given year |
|---|---|
| 0 | 144 |
| 1 | 91 |
| 2 | 32 |
| 3 | 11 |
| 4 | 2 |

There are $n = 280$ corps years and 196 deaths. So, there is an average of $\lambda = \frac{196}{280}$.

How does this distribution compare to the Poisson approximation?





## Example

☞ We compute the probabilities using the Poisson approximation with average $\lambda = \frac{196}{280} = 0.7$.

The random variable $X$ counts deaths in a corp in a year.

$$\begin{aligned}\mathbf{P}\{X = 0\} &= e^{-0.7} \approx 0.497 \\ \mathbf{P}\{X = 1\} &= e^{-0.7}(0.7) \approx 0.348 \\ \mathbf{P}\{X = 2\} &= e^{-0.7}(\frac{0.7^2}{2}) \approx 0.122 \\ \mathbf{P}\{X = 3\} &= e^{-0.7}(\frac{0.7^3}{3!}) \approx 0.028 \\ \mathbf{P}\{X = 4\} &= e^{-0.7}(\frac{0.7^4}{4!}) \approx 0.005\end{aligned}$$

## Example

☞ Comparing actual deaths from mule strikes (per corps) with Poisson prediction.

| # Deaths | # corps with $x$ deaths in a given year | Poisson prediction |
|---|---|---|
| 0 | 144 | 139 |
| 1 | 91 | 97 |
| 2 | 32 | 34 |
| 3 | 11 | 8 |
| 4 | 2 | 1 |





## Example

### Example

William Feller discusses the statistics of bomb strikes in an area in the south of London during the Second world war. The area in question was divided into $24 \times 24 = 576$ small areas. There were 537 hits. The number of times an area of hit was as follows:

| # of strikes | # areas with $x$ strikes |
|---|---|
| 0 | 229 |
| 1 | 211 |
| 2 | 93 |
| 3 | 35 |
| 4 | 7 |
| $\geq 5$ | 1 |

Assuming the hits were purely random, use the Poisson approximation to find the probability that a given square would have exactly $k$ hits.





## Example

☞ We compute the probabilities using the Poisson approximation with average $\lambda = \frac{537}{596}$.

The random variable $X$ counts bomb strikes in an area.

$$\begin{aligned}
\mathbf{P}\{X = 0\} &= e^{-\lambda} \approx 0.406 \\
\mathbf{P}\{X = 1\} &= e^{-\lambda}(\lambda) \approx 0.366 \\
\mathbf{P}\{X = 2\} &= e^{-\lambda}\left(\frac{\lambda^2}{2}\right) \approx 0.165 \\
\mathbf{P}\{X = 3\} &= e^{-\lambda}\left(\frac{\lambda^3}{3!}\right) \approx 0.05 \\
\mathbf{P}\{X = 4\} &= e^{-\lambda}\left(\frac{\lambda^4}{4!}\right) \approx 0.011 \\
\mathbf{P}\{X \geq 5\} &= 1 - \mathbf{P}\{X < 5\} \approx 0.0024
\end{aligned}$$





## Example

☞ Comparing actual strikes with Poisson prediction.

| # of strikes | # areas with $x$ strikes | Poisson prediction |
|---|---|---|
| 0 | 229 | 234 |
| 1 | 211 | 211 |
| 2 | 93 | 95 |
| 3 | 35 | 29 |
| 4 | 7 | 6 |
| $\geq 5$ | 1 | 1 |