Math 425
Introduction to Probability
Lecture 16

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Bernoulli Trials

Definition

By a Bernoulli trials process, we mean a sequence of trials (repetitions of an experiment) satisfying the following

1. Only two possible mutually exclusive outcomes on each trial.
   One arbitrarily called success and the other failure.
2. The probability of success on each trial is the same for each trial.
3. The trials are independent.

Examples of Bernoulli Trials

Examples. The following are examples of Bernoulli trials:

- Flip a coin (heads, tails),
- Each computer chip in a production line tested (chip passes test, fails test),
- Rolling a pair of dice for “snake-eyes" (double ones, any other value),
- A patient is prescribed a drug treatment (cured, not cured).
- A monkey types the complete works of Shakespeare (success, failure).

Random Variables and Bernoulli trials

☞ There are several discrete random variables associated with counting various events in a Bernoulli trials process.

- Binomial random variable: the number of successes in n trials.
  ☛ The possible values are integers between 0 and n.
- Geometric random variable: the number of trials until the first success.
  ☛ The possible values are all nonnegative integers (where the value 0 means success never occurs).
- Negative binomial random variable: the number of trials until k successes.
  ☛ The possible values are all nonnegative integers (where the value 0 means success never occurs).

Counting success

☞ In many applications we perform a fixed number n of Bernoulli trials. We would like to know the probability of k successes for various values $k \leq n$.

- A coin is flipped n times. What is the chance of exactly k heads?
- You have n widgets. What is the chance that k are defective?
- You type a page of n symbols. What is the chance of k errors?
- You place n bets on red in roulette. What is the chance you win k times?
- A gene consists of n base pairs. What is the chance that there are k mutations?

☞ The underlying problem in each case is the same. The only relevant things that change are the number of trials n, the probability of success p and the number k of successes.

Binomial Random Variable

☞ Consider a Bernoulli trials process consisting of n trials, with probability of success p and failure $q = 1 - p$.
Let B be the random variable counting the number of successes.

For $0 \leq k \leq n$ there are

$$\binom{n}{k}$$

sequences of length n consisting of exactly k successes, and each has the same probability

$$p^k q^{n-k}.$$

So,

$$\mathbf{P}\{B = k\} = \binom{n}{k} p^k q^{n-k} \qquad 0 \leq k \leq n.$$

Binomial Random Variable

Definition

Let $0 < p < 1$ and $q = 1 - p$ and $k > 0$. We say a random variable B is a binomial random variable (with parameters p and n) if

$$\mathbf{P}\{B = k\} = \binom{n}{k} p^k q^{n-k} \qquad 0 \leq k \leq n.$$

A random variable which counts the number of successes in a sequence of n Bernoulli trials is a binomial random variable.

Binomial Random Variable

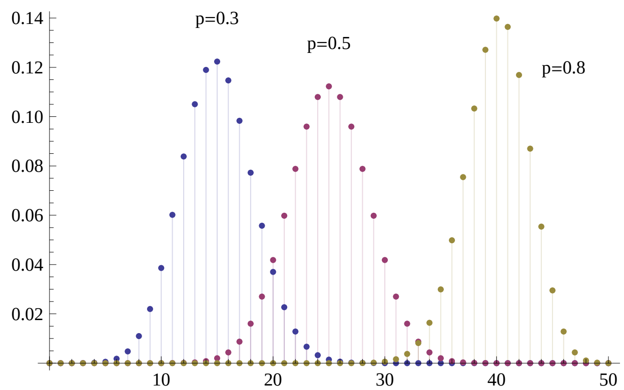
☞ Let B be a binomial random variable with parameters p and n.
Since the events $\{B = k\}$ are mutually exclusive

$$\begin{aligned}
\mathbf{P}\{0 \leq B \leq n\} &= \sum_{k=0}^{n} \mathbf{P}\{B = k\} \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \\
&= (p+q)^n = 1
\end{aligned}$$

The last line is the Binomial Theorem, Ross p. 8.

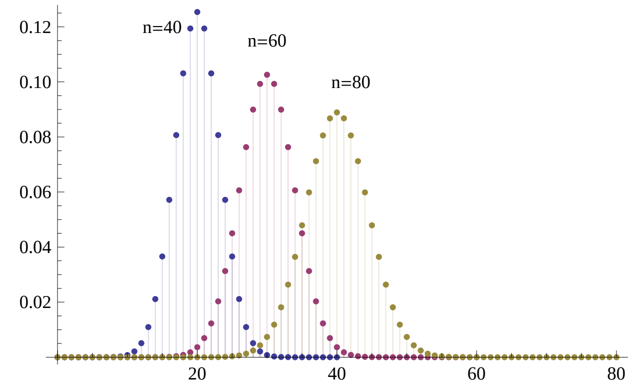
Binomial distribution

☞ Three distributions with fixed $n = 50$ and varying $p = 0.3, 0.5, 0.8$.



Binomial distribution

☞ Three distributions with varying $n = 40, 60, 80$ and fixed $p = 0.5$.



Mean and Variance

Theorem

Let B be a binomial random variable with parameter p, where $0 < p < 1$.

The mean and variance of B are given by

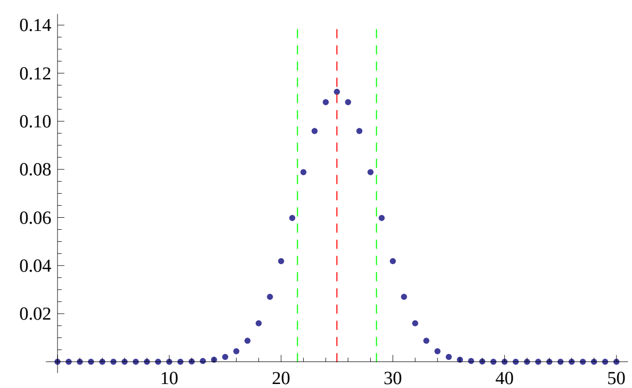
$$E[B] = np$$

$$Var(B) = np(1-p)$$

Binomial distribution

☞ Binomial random variable B with parameters $p = \frac{1}{2}$ and $n = 50$.

$E[B] = 25 \qquad Var(B) = \frac{50}{4} \qquad SD(B) \approx 3.5 \qquad \mathbf{P}\{22 \le B \le 28\} \approx 0.68.$



Note that the distribution is close to symmetric around its peak value.

Binomial distribution

☞ Statistics for a Binomial random variables with $n = 50$ and varying p.

p	$\mu = E[B]$	$Var(B)$	$\sigma = SD(B)$	$\mathbf{P}\{\mu - \sigma \leq B \leq \mu + \sigma\}$
0.1	5	4.5	2.12	0.766
0.25	12.5	9.375	3.06	0.745
0.5	25	12.5	3.5	0.678
0.75	37.5	9.375	3.06	0.738
0.9	45	4.5	2.12	0.766

Binomial distribution

Example. Acme Widgets sells their widgets in packs of 5 and guarantees that no more than one widget is defective. If a widget is defective with probability $p = 0.1$ what is the probability that a pack will be returned?

Solution. Let B be the binomial random variable counting defective widgets in a pack of 5. Then,

$$\begin{aligned} 1 - \mathbf{P}\{B \leq 1\} &= 1 - \mathbf{P}\{B = 0\} - \mathbf{P}\{B = 1\} \\ &= 1 - \binom{5}{0}(0.9)^5 - \binom{5}{1}(0.1)(0.9)^4 \\ &\approx 0.081. \end{aligned}$$

Discrete Random Variable

☞ A random variable X is discrete if there is an enumeration of all possible values X can take: $y_1, y_2, y_3, \ldots$, so that whenever $\mathbf{P}\{X = y\} > 0$ there is some i with $y = y_i$.

☞ We want to define the expected value of a discrete random variable as

$$E[X] = \sum_i y_i \cdot \mathbf{P}\{X = y_i\},$$

where $y_1, y_2, \ldots$ lists the possible values X can take.

☠ Warning: We need to make sure this makes sense for infinite discrete random variables!!

Discrete Random Variable

☞ If X takes only finitely many possible values $y_1, y_2, \ldots, y_n$, then

$$E[X] = \sum_i y_i \cdot \mathbf{P}\{X = y_i\}$$

always is well defined.

☞ If X can take infinitely many possible values $y_1, y_2, \ldots$, then

$$E[X] = \sum_i y_i \cdot \mathbf{P}\{X = y_i\}$$

may not be well defined because either (i) it does not converge or (ii) its value depends on the order of the summation.

Discrete Random Variable

Example. Let X be the random variable with the following probability mass function:

$$p_X(n) = \frac{c}{n^2} \qquad \text{where } c = \frac{6}{\pi^2} \text{ and } n \geq 1,$$

so that

$$\sum_n p_X(n) = c \sum_n \frac{1}{n^2} = c \cdot \frac{\pi^2}{6} = 1.$$

However,

$$E[X] = \sum_n n \cdot p_X(n) = c \sum_n \frac{1}{n} = \infty$$

The sum is the harmonic series, which diverges.

Discrete Random Variable

Example. Let X be the random variable with the following probability mass function:

$$p_X((-1)^{n+1} n) = \frac{c}{n^2} \qquad \text{where } c = \frac{6}{\pi^2} \text{ and } n \geq 1,$$

So,

$$E[X] = \sum_n (-1)^{n+1} n \cdot p_X(n) = c \sum_n \frac{(-1)^{n+1}}{n} = c \ln 2$$

The problem is that this series is only conditionally convergent, and what it converges to depends upon the order of the summation:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \quad < \quad \frac{5}{6}$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots \quad \geq \quad \frac{5}{6}$$

Discrete Random Variable

Definition

Let X be a discrete random variable which takes possible values $y_1, y_2, \ldots$ and has probability mass function p_X.

The expected value (or mean) of X is defined by

$$E[X] = \sum_i y_i \cdot p_X(y_i)$$

provided this sum converges absolutely:

$$\sum_i |y_i| \cdot p_X(y_i) \text{ converges.}$$

Geometric Random Variable

☞ Consider a Bernoulli trials process, with probability of success p and failure $q = 1 - p$, which we continue until the first success.

Let T be the number of trials up to and including the first success. Then

$$\mathbf{P}\{T = n\} = q^{n-1} p \qquad \text{when } n \geq 1$$

☞ Since the events $\{T = n\}$ are mutually exclusive

$$\mathbf{P}\{T \geq 1\} = \sum_{n=0}^{\infty} q^n p = \frac{p}{1-q} = \frac{p}{p} = 1,$$

and so

$$\mathbf{P}\{T = 0\} = 1 - \mathbf{P}\{T \geq 1\} = 1 - 1 = 0$$

$\{T = 0\}$ when all trials result in failure.

Geometric Random Variable

Definition

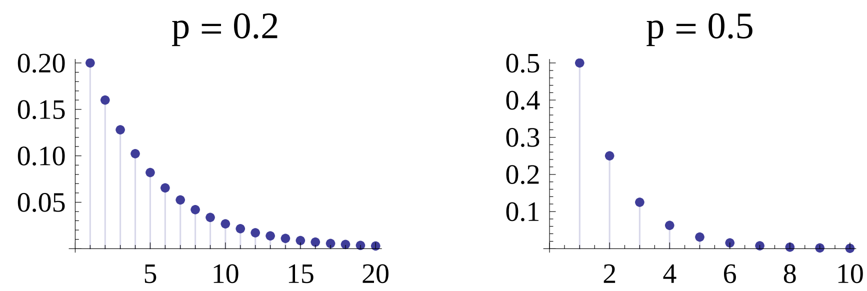
Let $0 < p < 1$ and $q = 1 - p$. We say a random variable T is a geometric random variable (with parameter p) if

$$\mathbf{P}\{X = n\} = q^{n-1}p \qquad n = 1, 2, 3, \ldots$$

A random variable which counts the number of trials until the first success in a Bernoulli trials process is a geometric random variable.

Geometric distributions

☞ Distributions for geometric random variables with $p = 0.2, 0.5$.



☞ Note that the most probable value is $\{T = 1\}$, and the values decrease rapidly as n increases.

Mean and Variance

☞ The mean of a geometric random variable T with parameter p is

$$E[T] = \sum_{n=1}^{\infty} n \cdot (1-p)^{n-1} p,$$

which converges absolutely.

Theorem

Let T be a geometric random variable with parameter p, where $0 < p < 1$. The mean and variance of T are

$$E[T] = \frac{1}{p}$$

$$Var(T) = \frac{1-p}{p^2}$$

Values for some distributions

p	$\mu = E[T]$	$Var(T)$	$\sigma = SD(T)$	$\mathbf{P}\{\mu - \sigma \le T \le \mu + \sigma\}$
$\frac{1}{6}$	6	30	5.477	0.865
$\frac{1}{4}$	4	12	3.464	0.867
$\frac{1}{3}$	3	6	2.45	0.868
$\frac{1}{2}$	2	2	1.414	0.875
$\frac{2}{3}$	1.5	0.75	0.866	0.889
$\frac{3}{4}$	1.33	0.444	0.667	0.9375

Examples

Example. A lottery chooses a number between 1 and 100 at random each week. What is the expected number of weeks between successive draws of the number 50?

Solution Let T be the geometric random variable which counts the weeks between successive draws of 50.

Since $p = 0.01$, the expected number of weeks is

$$E[T] = \frac{1}{p} = 100.$$

Negative Binomial Random Variable

☞ Consider a Bernoulli trials process, with probability of success p and failure $q = 1 - p$, which we continue until k successes (where $k > 0$).

Let X_k be the number of trials up to and including the k success. For $n \geq k$, $X_k = n$ exactly when there are $k - 1$ successes in the first $n - 1$ trials, and the last trial is a success.

There are

$$\binom{n-1}{k-1}$$

such sequences of length n and each has the same probability

$$p^k q^{n-k}.$$

Therefore

$$\mathbf{P}\{X_k = n\} = \binom{n-1}{k-1} p^k q^{n-k} \qquad n = k, k+1, k+2, \ldots$$

Negative Binomial Random Variable

Definition

Let $0 < p < 1$ and $q = 1 - p$ and $k > 0$. We say a random variable X is a negative binomial random variable (with parameters p and k) if

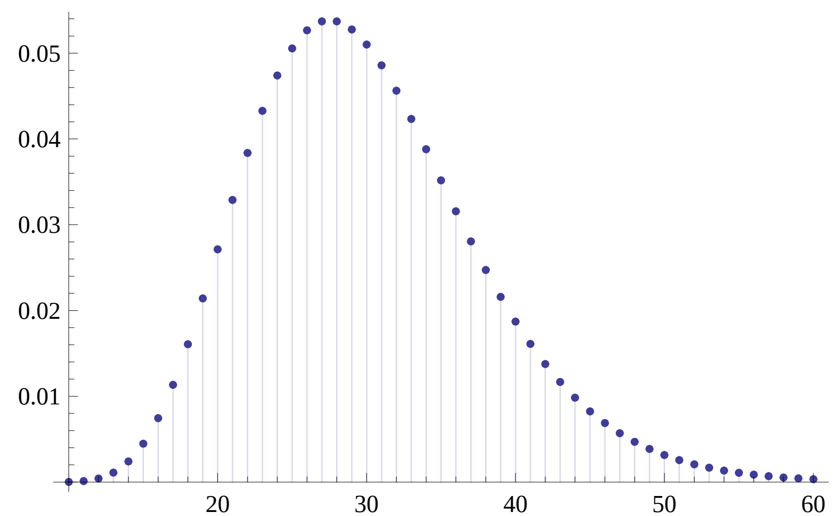
$$\mathbf{P}\{X = n\} = \binom{n-1}{k-1} p^k q^{n-k} \qquad n = k, k+1, k+2, \ldots$$

A random variable which counts the number of trials until the kth success in a Bernoulli trials process is a negative random variable.

When $k = 1$, this is the same as a geometric random variable.

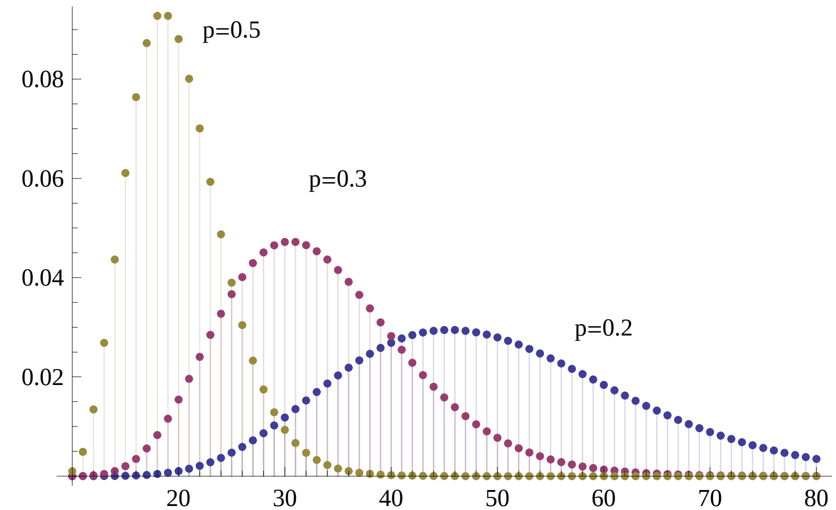
Negative Binomial distribution

☞ Distribution for negative binomial random variables with $p = \frac{1}{3}$ and $k = 10$.



Negative Binomial distribution

☞ Distribution for negative binomial random variables with $p = 0.3, 0.5$ and $k = 10$.



Mean and Variance

☞ The mean of a negative random variable X with parameters p and k is

$$E[X] = \sum_{n=k}^{\infty} n \cdot \binom{n}{k} p^k (1-p)^{n-k},$$

which converges absolutely.

Theorem

Let X be a negative binomial random variable with parameters p, where $0 < p < 1$ and $k > 0$. The mean and variance of X are

$$E[X] = \frac{k}{p}$$

$$Var(X) = \frac{k(1-p)}{p^2}$$

Values for some distributions

☞ Negative Binomial Random Variable with $k = 2$ and various p

p	$\mu = E[X]$	$Var(X)$	$\sigma = SD(X)$	$\mathbf{P}\{\mu - \sigma \le X \le \mu + \sigma\}$
$\frac{1}{6}$	12	60	7.75	0.796
$\frac{1}{4}$	8	24	4.9	0.958
$\frac{1}{3}$	6	12	3.46	0.941
$\frac{1}{2}$	4	4	2	0.891
$\frac{2}{3}$	3	1.5	1.225	0.889
$\frac{3}{4}$	2.66	0.889	0.942	0.844

Example

Example. What is the probability of k successes before ℓ failures in a Bernoulli trials process with probability p?

Solution. If k successes comes before ℓ failures, then the k success must occur on some trial n with $k \le n < k + \ell$.

Let X count the number of trials before the kth success. Then

$$\mathbf{P}\{k \le X < k + \ell\} = \sum_{n=k}^{k+\ell-1} \binom{n-1}{k-1} p^k (1-p)^{n-k}.$$