Math 425
Introduction to Probability
Lecture 10

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Multiplication Rule for 2 events

Lemma (Multiplication Rule)

For any events E and F (where $\mathbf{P}(F) > 0$),

$$\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F \mid E)$$

Multiplication Rule for 3 events

☞ We can extend the Multiplication Rule to three events.

Lemma

For any events E, F, G (provided $\mathbf{P}(E \cap F \cap G) > 0$)

$$\mathbf{P}(E \cap F \cap G) = \mathbf{P}(E) \cdot \mathbf{P}(F \mid E) \cdot \mathbf{P}(G \mid E \cap F)$$

Proof. Use the Mutiplication Rule twice,

$$\begin{aligned} \mathbf{P}(E \cap F \cap G) &= \mathbf{P}(E \cap F) \cdot \mathbf{P}(G \mid E \cap F) \\ &= \mathbf{P}(E) \cdot \mathbf{P}(F \mid E) \cdot \mathbf{P}(G \mid E \cap F) \end{aligned}$$

We need $\mathbf{P}(E \cap F \cap G) \neq 0$ to ensure the conditional probabilities exist.

Example: 3 events

Example

An urn is filled with 6 red balls, 5 blue balls, and 4 green balls. Three balls chosen at random are removed from the urn.

What is the probability that the balls are of the same color?

☞ We are interested in the events (where $i = 1, 2, 3$)

- R_i: ith ball drawn is red,
- B_i: ith ball drawn is blue,
- G_i: ith ball drawn is green.
- C: three balls are the same color.

Example – continued

☞ Urn: 6 red, 5 blue, and 4 green. Use the Multiplication Rule,

$$\begin{aligned} \mathbf{P}(R_1 \cap R_2 \cap R_3) &= \mathbf{P}(R_1) \cdot \mathbf{P}(R_2 \mid R_1) \cdot \mathbf{P}(R_3 \mid R_1 \cap R_2) \\ &= \frac{6}{15} \cdot \frac{5}{14} \cdot \frac{4}{13} = \frac{4}{91} \end{aligned}$$

$$\begin{aligned} \mathbf{P}(B_1 \cap B_2 \cap B_3) &= \mathbf{P}(B_1) \cdot \mathbf{P}(B_2 \mid B_1) \cdot \mathbf{P}(B_3 \mid B_1 \cap B_2) \\ &= \frac{5}{15} \cdot \frac{4}{14} \cdot \frac{3}{13} = \frac{2}{91} \end{aligned}$$

$$\begin{aligned} \mathbf{P}(G_1 \cap G_2 \cap G_3) &= \mathbf{P}(G_1) \cdot \mathbf{P}(G_2 \mid G_1) \cdot \mathbf{P}(G_3 \mid G_1 \cap G_2) \\ &= \frac{4}{15} \cdot \frac{3}{14} \cdot \frac{2}{13} = \frac{4}{455} \end{aligned}$$

Since these events are mutually exclusive,

$$\mathbf{P}(C) = \frac{4}{91} + \frac{2}{91} + \frac{4}{455} = \frac{34}{455} \approx 0.0747.$$

General multiplication Rule

☞ The Multiplication rule is the probabilistic version of the product rule for counting.

Theorem (Generalized Multiplication Rule)

Let $E_1, E_2, \ldots, E_n$ be any events such that

$$\mathbf{P}(E_1 \cap E_2 \cap \cdots \cap E_n) > 0.$$

Then

$$\begin{aligned} \mathbf{P}(E_1 \cap E_2 \cap \ldots \cap E_n) &= \mathbf{P}(E_1) \cdot \mathbf{P}(E_2 \mid E_1) \cdot \mathbf{P}(E_3 \mid E_1 \cap E_2) \cdots \\ &\quad \cdots \mathbf{P}(E_n \mid E_1 \cap E_2 \cap \ldots \cap E_{n-1}). \end{aligned}$$

(See Ross p. 71.)

Example: 6 events

Example

In Pick-Six Lottery: A person purchases a ticket, and can choose 6 distinct numbers in the set $\{1, 2, 3, \ldots, 49\}$.

Later a Lottery Machine picks 6 distinct numbers at random in the set $\{1, 2, 3, \ldots, 49\}$.

A winning ticket is one which matches the six numbers chosen by the Machine (in any order of selection).

☞ What are the odds of winning with one ticket?

Example – solution

Solution. We solved this before (Lecture 5) by counting using the Product Rule for counting. We conditionalize here.

☞ Let E_i be the event that there are i matches. We want to compute

$$\begin{aligned} \mathbf{P}(E_6) &= \mathbf{P}(E_1 \cap E_2 \cap \ldots \cap E_6) \\ &= \mathbf{P}(E_1) \cdot \mathbf{P}(E_2 \mid E_1) \cdots \mathbf{P}(E_6 \mid E_1 \cap \ldots \cap E_5) \\ &= \frac{6}{49} \cdot \frac{5}{48} \cdot \frac{4}{47} \cdot \frac{3}{46} \cdot \frac{2}{45} \cdot \frac{1}{44} \\ &= \frac{1}{13{,}983{,}816} \end{aligned}$$

Dependence

☞ Sometimes, changes in the conditions of an experiment change the probability of some outcomes.

Example. An urn has 7 red balls and 5 blue balls. The balls are well mixed. A ball is drawn, its color is noted and put aside.

Compare the probability that the second ball is red (R_2) given that the first drawn ball is red (R_1) versus that it is blue (R_1^c).

$$\mathbf{P}(R_2 \mid R_1) = \frac{6}{11} \qquad \mathbf{P}(R_2 \mid R_1^c) = \frac{7}{11}$$

The probability that the second ball is red is

$$\begin{aligned} \mathbf{P}(R_2) &= \mathbf{P}(R_2 \mid R_1) \cdot \mathbf{P}(R_1) + \mathbf{P}(R_2 \mid R_1^c) \cdot \mathbf{P}(R_1^c) \\ &= \frac{6}{11} \cdot \frac{7}{12} + \frac{7}{11} \cdot \frac{5}{12} = \frac{7}{12} \end{aligned}$$

Independence

☞ Sometimes, changes in the conditions of an experiment have no effect on the probability of some outcomes.

Example. An urn has 7 red balls and 5 blue balls. The balls are well mixed. A ball is drawn, its color is noted and returned to the urn, which is again well mixed.

Compare the probability that the second ball is red (R_2) given that the first drawn ball is red (R_1) versus that it is blue (R_1^c).

$$\mathbf{P}(R_2 \mid R_1) = \frac{7}{12} \qquad \mathbf{P}(R_2 \mid R_1^c) = \frac{7}{12}$$

The probability that the second ball is red is

$$\begin{aligned} \mathbf{P}(R_2) &= \mathbf{P}(R_2 \mid R_1) \cdot \mathbf{P}(R_1) + \mathbf{P}(R_2 \mid R_1^c) \cdot \mathbf{P}(R_1^c) \\ &= \frac{7}{12} \cdot \frac{7}{12} + \frac{5}{12} \cdot \frac{7}{12} = \frac{7}{12} \end{aligned}$$

Independence

☞ Suppose E and F are events with $\mathbf{P}(E \mid F) = \mathbf{P}(E \mid F^c)$.

☞ By the Partition Rule

$$\begin{aligned} \mathbf{P}(E) &= \mathbf{P}(E \mid F)\mathbf{P}(F) + \mathbf{P}(E \mid F^c)\mathbf{P}(F^c) \\ &= x \cdot \left[\mathbf{P}(F) + \mathbf{P}(F^c)\right] = x \qquad \text{where } x \text{ is } \mathbf{P}(E \mid F) \text{ or } \mathbf{P}(E \mid F^c) \end{aligned}$$

So, $\mathbf{P}(E) = \mathbf{P}(E \mid F)$ and $\mathbf{P}(E) = \mathbf{P}(E \mid F^c)$.

☞ By the Multiplication Rule

$$\mathbf{P}(E \cap F) = \mathbf{P}(E \mid F) \cdot \mathbf{P}(F) = \mathbf{P}(E) \cdot \mathbf{P}(F).$$

So, $\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F)$

Independence and the Product Rule

Definition (Product Rule)

Events E and F and independent if and only if

$$\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F).$$

Equivalently, E and F are independent if and only if

$$\mathbf{P}(E \mid F) = \mathbf{P}(E) = \mathbf{P}(E \mid F^c)$$

Events which are not independent are said to be dependent.

Proof of Equivalence

☞ We have already shown the

$$\mathbf{P}(E \mid F) = \mathbf{P}(E \mid F^c) \quad \Rightarrow \quad \mathbf{P}(E \mid F) = \mathbf{P}(E) \text{ and } \mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F).$$

☞ Conversely, suppose $\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F)$.
By the Conditioning Rule

$$\mathbf{P}(E \mid F) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)} = \frac{\mathbf{P}(E) \cdot \mathbf{P}(F)}{\mathbf{P}(F)} = \mathbf{P}(E).$$

By the Conditioning and Partition Rules,

$$\begin{aligned} \mathbf{P}(E \mid F^c) &= \frac{\mathbf{P}(E \cap F^c)}{\mathbf{P}(F^c)} \\ &= \frac{\mathbf{P}(E) - \mathbf{P}(E \cap F)}{\mathbf{P}(F^c)} \qquad \text{since } \mathbf{P}(E) = \mathbf{P}(E \cap F) + \mathbf{P}(E \cap F^c) \\ &= \frac{\mathbf{P}(E) \cdot (1 - \mathbf{P}(F))}{\mathbf{P}(F^c)} = \mathbf{P}(E) \end{aligned}$$

Example

Example

A standard 52 card deck is well shuffled. Are the following events independent:

- E: Draw a ♠,
- F: Draw an ace?

Solution. E and F are independent.

$$\mathbf{P}(E \mid F) = \frac{1}{4} \qquad \mathbf{P}(E \mid F^c) = \frac{12}{48} = \frac{1}{4}$$

Example

Example

Three dice are thrown. Are the following events independent:

- E_6: Throw a six on at least one die,
- S_{16}: The sum of the dice is 16?

Solution. E and F are dependent.

$$\mathbf{P}(S_{16} \mid E_6) > 0 \qquad \mathbf{P}(S_{16} \mid E_6^c) = 0$$

Challenge: verify $\mathbf{P}(S_{16} \mid E_6) = \frac{6}{91}$.

Extended Product Rule

Definition (Extended Product Rule)

The events $E_1, E_2, \ldots$ (possibly infinitely many events) are independent if and only if

$$\mathbf{P}(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_n}) = \mathbf{P}(E_{i_1}) \cdot \mathbf{P}(E_{i_2}) \cdots \mathbf{P}(E_{i_n})$$

for any finite subset of indices $i_1, i_2, \ldots, i_n$.

Equivalently, the events $E_1, \ldots, E_2, \ldots$ are independent if and only if

$$\mathbf{P}(E_{i_n} \mid E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{n-1}}) = \mathbf{P}(E_{i_n}),$$

for any finite ($n \geq 2$) subset of indices $i_1, i_2, \ldots, i_n$.

Example

Example

A sequence of fair coins is flipped n times, and each outcome is equiprobable. Let E_i be the event that the ith flip is heads.
Are the events $E_1, E_2, \ldots, E_n$ equiprobable?

Solution. They are independent: Fix any $k+1$ events. Then

$$\mathbf{P}(E_{i_{k+1}} \mid E_{i_1} \cap \cdots \cap E_{i_k}) = \frac{2^{n-k-1}}{2^{n-k}} = \frac{1}{2}$$

$$\mathbf{P}\left(E_{i_{k+1}}\right) = \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

Conditional Independence

☞ It is possible that two events E and F are not independent, but they become so on the assumption that a third event G occurs.

Definition

Events E and F are conditionally independent given G if

$$\mathbf{P}(E \cap F \mid G) = \mathbf{P}(E \mid G) \cdot \mathbf{P}(F \mid G).$$

Equivalently,

$$\mathbf{P}(E \mid F \cap G) = \mathbf{P}(E \mid G).$$

Proof of Equivalence

☞ Suppose $\mathbf{P}(E \mid F \cap G) = \mathbf{P}(E \mid G)$. Then,

$$\frac{\mathbf{P}(E \cap F \cap G)}{\mathbf{P}(F \cap G)} = \mathbf{P}(E \mid G) \quad \text{Conditioning Rule for } \mathbf{P}(E \mid F \cap G)$$

$$\mathbf{P}(E \cap F \cap G) = \mathbf{P}(E \mid G) \cdot \mathbf{P}(F \cap G)$$

$$\mathbf{P}(E \cap F \mid G) = \mathbf{P}(E \mid G) \cdot \mathbf{P}(F \mid G) \quad \text{divide both sides by } \mathbf{P}(G)$$

☞ Suppose $\mathbf{P}(E \cap F \mid G) = \mathbf{P}(E \mid G) \cdot \mathbf{P}(F \mid G)$. Then,

$$\frac{\mathbf{P}(E \cap F \cap G)}{\mathbf{P}(G)} = \mathbf{P}(E \mid G) \cdot \frac{\mathbf{P}(F \cap G)}{\mathbf{P}(G)} \quad \text{Conditioning Rule}$$

$$\frac{\mathbf{P}(E \cap F \cap G)}{\mathbf{P}(F \cap G)} = \mathbf{P}(E \mid G)$$

$$\mathbf{P}(E \mid F \cap G) = \mathbf{P}(E \mid G)$$

Example

Example

Suppose you roll a red and blue die. Consider the events

- L_2: lower score is 2,
- H_5: higher score is 5.
- D: one die is greater than 3 and one die is less than 3.

Then,

(a) L_2 and H_5 are not independent,

(b) L_2 and H_5 are conditionally independent given D.

Example

(a). L_2 and H_5 are not independent

$$\mathbf{P}(L_2 \cap H_5) = \frac{2}{36} = \frac{1}{18}$$

$$\mathbf{P}(L_2) \cdot \mathbf{P}(H_5) = \frac{9}{36} \cdot \frac{9}{36} = \frac{1}{16}$$

(b). L_2 and H_5 are conditionally independent given D

$$\mathbf{P}(L_2 \mid D) = \frac{1}{2}$$

$$\mathbf{P}(L_2 \mid H_5 \cap D) = \frac{1}{2}$$

Example

Example

Suppose you roll a red and blue die. Consider the events

- R_2: a 2 on the red die,
- B_2: a 2 on the blue die,
- D: one die is greater than 3 and one die is less than 3.

Then,

(a) R_2 and B_2 are independent,

(b) R_2 and B_2 are not conditionally independent given D.

Example

(a). R_2 and B_2 are independent:

$$\mathbf{P}(R_2 \cap B_2) = \frac{1}{36} = \mathbf{P}(R_2) \cdot \mathbf{P}(B_2).$$

(b). R_2 and B_2 are not conditionally independent given D:

$$\mathbf{P}(R_2 \mid D) = \frac{\mathbf{P}(R_2 \cap D)}{\mathbf{P}(D)} = \frac{\frac{3}{36}}{\frac{12}{36}} = \frac{1}{4}$$

$$\mathbf{P}(R_2 \mid B_2 \cap D) = 0$$

Example: Baseball

☞ Compare to the Problem of points, Example 3.4j of Ross, p. 95.

Example

The Cubs (!!) and White Sox are playing in the World Series. The Cubs win each game with probability 0.6 (independently of the games played). What is the probability that the Cubs win the Series. (The first team to win four games wins the series.)

Example: Baseball

Method 1. (Due to Fermat)

Since the first to four wins takes the Series, there are at most 7 games (4 Cub wins to 3 Sox wins). Fermat takes the sample space to be sequences of length 7:

$$(g_1, g_2, g_3, g_4, g_5, g_6, g_7) \quad \text{where } g_i = W, L$$

Any sequence with 4 Ws is a Cub win, otherwise it is a Sox win (there are at least 4 Ls).

Not all sequences represent actual outcomes, nor are all sequences equally likely.

☞ Why does it not matter to extend a real Series play with phantom games?

The phantom games do not change the probability: If the Cubs win the series in four games, they win every extension with phantom games as well.

Example: Baseball

☞ If X is an outcome in the sample space and has k Ws (so, $7 - k$ Ls), then

$$\mathbf{P}(X) = (0.6)^k (0.4)^{7-k}$$

☞ In this sample space, any sequence with at least 4 Ws is a Cubs Series win.

$$\mathbf{P}(\text{Cubs win}) = \sum_{n=4}^{7} \binom{7}{n} \cdot (0.6)^n (0.4)^{7-n} \approx 0.71.$$

Example: Baseball

Method 2. (Due to Pascal)

This method allows us to treat the sample space as sequences whose length is at most 7, with 4 Ws or 4 Ls. Let

- $W_{n,m}$ be the event that the Cubs win the series when they have n wins and the Sox have m wins. (where $n, m \leq 4$)

At the start of the Series, we want to compute $W_{0,0}$.

☞ $W(0,0)$ can be determined from the following 3 conditions:

(a) $\mathbf{P}(W_{4,n}) = 1$, where $n \leq 3$,

(b) $\mathbf{P}(W_{n,4}) = 0$ where $m \leq 3$,

(c) When the teams have played $n + m$ games, and both $n, m < 4$, then

$$\mathbf{P}(W_{n,m}) = 0.6 \cdot \mathbf{P}(W_{n+1,m}) + 0.4 \cdot \mathbf{P}(W_{n,m+1}).$$

The next game is either a Cubs win or Sox win.

☞ For example,

$$\mathbf{P}(W_{3,3}) = p \quad \mathbf{P}(W_{3,2}) = 0.6 + 0.4 \cdot 0.6 = 0.84 \mathbf{P}(W_{2,3}) = 0.6^2 = 0.36$$