Math 425
Introduction to Probability
Lecture 1

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

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Course Data

- **Text.** *A First Course in Probability* by Sheldon Ross (7th ed.)
  We will cover Chapters 1-8.
- **Office.** 1842 East Hall
- **Office hours.** 10-11, 1-2 (M,W,F) and by appointment

Web page

- **Web.** `http://www-personal.umich.edu/~kaharri/425/`
  Linked off CTools
- All course information will be posted on the web page:
  - daily schedule,
  - notes,
  - homework assignments,
  - study guide for exams,
  - class announcements.

Grading

- Grades will be based on the following:
  1. Weekly Homework : 25%
  2. Midterm 1 (Feb 6) : 20%
  3. Midterm 2 (March 18) : 25%
  4. Final (April 27) : 30%
- Homework is assigned on Wednesday (including today!), and due the following Wednesday. (Except today's assignment, which is due in two weeks.)

Probability vs. Statistics

Probability. In Probability, we consider an experiment before it is performed; and we deduce the probability (likelihood) of various possible outcomes of an experiment.

- Probability is the mathematical theory of measuring uncertainty.

Statistics. In Statistics, we infer things from the observed outcomes of an experiment already performed.

- Statistics is the mathematical theory of making decisions in the face of uncertainty.

This course will be about Probability, not Statistics.

Chapter 1

Chapter 1. We learn how to measure uncertainty, by first learning how to count.

Addditive Counting Principle

Principle (Sum Rule)

Suppose we have k experiments where each has $n_1, n_2, \ldots, n_k$ possible outcomes, respectively, and no two experiments have the same outcome.

Then there are $n_1 + n_2 + \ldots + n_k$ possible outcomes (given only that some one of the k experiments was perfomed).

Example: Addditive Counting Principle

Example. We have a standard deck of 52 playing cards.

Experiment 1: Select a card for a suit.
Outcomes: $\heartsuit, \diamondsuit, \clubsuit, \spadesuit$.

Experiment 2: Select a card for a value.
Outcomes: $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$.

Experiment 3: Select a card for a color.
Outcomes: red, black.

Question. How many possible outcomes are there, given only one experiment is performed?

Answer. 19 outcomes by the Sum Rule.

Sequential Counting Principle

Principle (Product Rule)

Suppose a procedure can be broken down into two successive experiments, where the first experiment has m possible outcomes and for each of these outcomes there are n possible outcomes of the second experiment.

Then there are $m \cdot n$ possible outcomes of the procedure.

Example 1: Sequential Counting Principle

Example. We have a standard deck of 52 playing cards. The procedure is to take a card from the deck, replace the card and shuffle the deck, then take a second card from the deck. An experiment consists of choosing a card, and its outcome is the card (suit and value).

Question. How many possible outcomes of this procedure?

Answer. $52 \cdot 52 = 2704$ outcomes by the Product Rule.

- There are 52 possible outcomes for the first experiment.
- For each outcome of the first experiment, there are 52 outcomes of the second experiment.

Example 2: Sequential Counting Principle

Example. We have a standard deck of 52 playing cards. The procedure is to take a card from the deck, put it aside, then take a second card from the deck. An experiment consists of choosing a card, and its outcome is the card (suit and value).

Question. How many possible outcomes of this procedure?

Answer. $52 \cdot 51 = 2652$ outcomes by the Product Rule.

- There are 52 possible outcomes for the first experiment.
- For each outcome of the first experiment, there are 51 possible outcomes of the second experiment, since only 51 cards remain in the deck.

Example 3: Sequential Counting Principle

Example. We have a standard deck of 52 playing cards. The procedure is to take a card from the deck, put it aside, then take a second card from the deck. An experiment consists of choosing a card, and its outcome is the suit of the card.

Question. How many possible outcomes of this procedure?

Answer. $4 \cdot 4 = 16$ possible outcomes by the Product Rule.

- There are 4 possible outcomes for the first experiment.
- For each outcome of the first experiment, there are still cards of all four suits, so there are 4 possible outcomes of the second experiment.

Proof of Product Rule

☞ For each outcome i of the first experiment, there are n possible outcomes for the second experiment. List these as

$$(i,1) \quad (i,2) \quad (i,3) \quad \dots \quad (i,n)$$

☞ Lets call this Experiment i.
(We run Experiment i only when the first experiment has outcome i, by running the second experiment.)

Proof of Product Rule

☞ We have m Experiments, each with different outcomes

$$\begin{array}{cccccc} \text{Experiment } 1 & (1,1) & (1,2) & (1,3) & \dots & (1,n) \\ \text{Experiment } 2 & (2,1) & (2,2) & (2,3) & \dots & (2,n) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{Experiment } m & (m,1) & (m,2) & (m,3) & \dots & (m,n) \end{array}$$

☞ By the Sum Rule there are $n + n + \ldots + n = m \cdot n$ total outcomes given we run only one Experiment.

☞ The outcome of this one Experiment is the same as the outcome of our original procedure.

Generalized Sequential Counting Principle

Principle (Generalized Product Rule)

Suppose a procedure can be broken down into k successive experiments, where the first experiment has n_1 possible outcomes, and for each of these outcomes there are n_2 possible outcomes of the second experiment, and for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and so on.

Then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ possible outcomes of the procedure.

Example 1: Sequential Counting Principle

Example. A coin is tossed k times, and each toss can have the outcome heads up or tails up. An experiment consists of a single toss of the coin.

Question. How many possible outcomes are there in k coin tosses?

Answer. 2^k outcomes by the Generalized Product Rule.

Example 2: Seating arrangements

Example. 3 boys and 3 girls are to sit in a row.

Question. How many seating arrangements are there?

Answer. There are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720$ arrangements.

- There are six experiments: Each experiment is to choose the next person from those who were not previously chosen. (This reduces the possible outcomes by one for the next choice to be made.)

Example 3: Seating arrangements

Example. 3 boys and 3 girls are to sit in a row.

Question. How many seating arrangements are there if the boys and the girls are each to sit together?

Answer. There are $(3! \cdot 3!) + (3! \cdot 3!) = 72$ arrangements.

- First, choose a seating order based on gender. There are two

  BBBGGG *GGGBBB*

- Second, choose the individuals, where the choice is constrainted by gender.

Example 4: Seating arrangements

Example. 3 boys and 3 girls are to sit in a row.

Question. How many seating arrangements are there if the boys and the girls must alternate?

Answer. There are $(3! \cdot 3!) + (3! \cdot 3!) = 72$ arrangements.

- First, choose a seating order based on gender. There are two

  BGBGBG *GBGBGB*

- Second, choose the individuals, where the choice is constrainted by gender.

Example 4: Seating arrangements

Example. 3 boys and 3 girls are to sit in a row.

Question. How many seating arrangements are there if only the boys must sit together?

Answer. There are $4 \times (3! \cdot 3!) = 144$ arrangements.

- First, choose a seating order based on gender. There are two

  BBBGGG *GBBBGG* *GGBBBG* *GGGBBB*

- Second, choose the individuals, where the choice is constrainted by gender.

Permutations

Definition
A permutation is an arrangement of objects in a certain order.

Example. There are 6 permutations of the three stooges.

$$(\text{Moe}, \text{Larry}, \text{Curly}) \quad (\text{Moe}, \text{Curly}, \text{Larry}) \quad (\text{Larry}, \text{Moe}, \text{Curly})$$
$$(\text{Larry}, \text{Curly}, \text{Moe}) \quad (\text{Curly}, \text{Moe}, \text{Larry}) \quad (\text{Curly}, \text{Larry}, \text{Moe})$$

Counting permutations

Notation $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$ (Factorial)

Fact. There are $n!$ permutations of n distinct objects.

Examples

- There are $3! = 6$ permutations of the three stooges.
- **Question.** How many different ways are there of shuffling a deck of cards?
  Answer. 52! (52 factorial – which is alot!)

Generalized permutation principle

Principle
The number of ways of choosing r distinct objects taken from of a set of n objects (where order matters) is

$$\frac{n!}{(n-r)!} = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$$

Example.

- How many 4 digit PIN numbers are there, if the digits must be distinct?
  $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ different 4 digit PIN numbers with distinct digits.
- How many 4 digit PIN numbers are there?
  $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ different 4 digit PIN numbers.

Example 1

Example. A typical basketball team carries 4 guards, 2 centers and 4 forwards.

Question. A coach must choose a starting lineup consisting of a point guard, shooting guard, small forward, power forward and center.
How many different starting lineups are there (assuming a guard can play either position and a forward can play either position)?

Answer. $(4 \cdot 3) \cdot (4 \cdot 3) \cdot 2 = 288$

- A starting lineup consists of

  SG PG SF PF C

- Choose each play by position: there are $4 \cdot 3$ ways to choose the guards, $4 \cdot 3$ ways to choose the forwards and 2 ways to choose the center.