MATH 425
Midterm 2
Winter, 2009

Name:

- You have **50 minutes** to complete your work.
- Show all work and make it clear what your answers are.
- You are permitted two 3x5 notecard. Otherwise, books, notes, calculators and computers are not permitted on this exam.

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problem	points	score
1	20	
2	20	
3	15	
4	10	
5	10	
6	10	
Total	90	

1. (20 points)
For each of the random variables X below, determine the type of distribution which best models X. Give the values of the parameters to the distribution you have chosen. State your assumptions you think relevant for your choice. Space is given for your answers on the next page.

a. In a class of 80 students, the professor calls on 2 students chosen at random for a recitation in each class period. There are 40 class periods in a term. Let X be the random variable which counts the number of times student s is called in a term.

b. A small college has found that on average only 1 in every 3 students accepted will actually attend. They accept 450 students for next year's class. Let X count the number of students who actually attend.

c. During the peak time of the Capricornids meteor shower, the average time between meteors is 4 minutes. Let X count the time (in hours) you wait until you observe 25 meteors.

d. Smith is playing craps and tossed a 5 on his first throw. He must continue tossing until either a 5 appears (and he wins) or a 7 appears (and he loses). Let X count the number of throws (not counting Smith's first throw) until a winner is determined.

1. Work space.

a.

X is counting the number of successes in a Bernoulli trials process, where each trial is a selection of a student for recitation, and $p = \frac{1}{80}$. Technically, X is binomial random variable with $n = 80$ and $p = \frac{1}{80}$. However, given the large number of trials, and the low probability, it is best modeled by a Poisson random variable with $\lambda = 1$. (This is the average number of times a student will be called in a term.)

I have assumed that each student is equally likely to be chosen for each recitation. This amounts to the assumption that each trial is independent. This assumes the professor is a bit absent-minded about who has been called during the term.

b.

X is counting the number of successes in a Bernoulli trials process, where each trial is that a student accepts or does not accept, and $p = \frac{1}{3}$. Technically, X is a binomial random variable with $n = 450$ and $p = \frac{1}{3}$. However, given the large number of trials, and the moderate probability, it is best modeled by a normal random variable.

The parameters are μ (the mean of the binomial random variable) and σ^2 (the variance of the binomial random variable):

$$\mu = 450 \cdot \frac{1}{3} = 150 \qquad \sigma^2 = 450 \cdot \frac{1}{3} \cdot \frac{2}{3} = 100.$$

The standard deviation $\sigma = 10$ is also acceptable as a parameter value (in place of σ^2).

I have assumed that the decision of a student to attend or not is independent of the decision of any other student.

c.

X is counting the waiting time in a Poisson process for 25 events, each meteor. This is modeled by a gamma distribution. Since X counts the time in hours until 25 meteors are observed, the parameter $\lambda = 15$ is the number of meteors per hour, and $\alpha = 25$ is the number of meteors.

I have assumed that the occurrence of meteors are independent events.

d.

X is counting the number of trials until the first success in a Bernoulli trials process, where each trial is a toss of the pair of die. So, X is a geometric random variable. A success in a trial is that a 5 or 7 appears. This happens with probability $p = \frac{10}{36} = \frac{5}{18}$.

I have assumed that Smith is not a die shark, so that each number is equally likely to appear on any throw.

2. (20 points)

a. Let X be a uniformly distributed random variable over $[0, 2]$. Find the probability density function for $Y = e^X$.

Theorem 5.7.1 applies here with $g(x) = e^x$ and $g^{-1}(y) = \log y$. For $1 \le y \le e^2$ (the possible values of Y)

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right) \cdot \frac{d}{dy} g^{-1}(y) \\ &= \frac{1}{2} \cdot \frac{1}{y} \\ &= \frac{1}{2y} \end{aligned}$$

So, the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{2y} & \text{if } 1 \le y \le e^2, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, you can prove this directly: for $1 \le y \le e^2$

$$\begin{aligned} F_Y(y) &= \mathbf{P}\{Y \le y\} \\ &= \mathbf{P}\{e^X \le y\} \\ &= \mathbf{P}\{X \le \log y\} \\ &= \int_0^{\log y} \frac{1}{2}\, dx \\ &= \frac{\log y}{2} \end{aligned}$$

The density is obtained by differentiating $F_Y(y)$, and gives the same result.

b. Let X be a random variable with $E[X] = 2$ and $Var(X) = 3$, and $Y = 3X + 4$. Find the expectation and variance of Y.

$$\begin{aligned} E[Y] &= E[3X + 4] = 3E[X] + 4 = 10 \\ Var(Y) &= Var(3X + 4) = 9Var(X) = 27 \end{aligned}$$

c. Let X be a normally distributed random variable with $\mu = 4$ and $\mathbf{P}\{X > 0\} = 0.7881$. Find the variance σ^2 for X.

$Z = \frac{X-\mu}{\sigma}$, where $\mu = E[X] = 4$ and σ is unknown, is a standard normal distribution. So,

$$\begin{aligned} 0.7881 &= \mathbf{P}\{X > 0\} \\ &= \mathbf{P}\{Z > \frac{0-4}{\sigma}\} \\ &= \mathbf{P}\{Z > -\frac{4}{\sigma}\} \\ &= \Phi(\frac{4}{\sigma}) \end{aligned}$$

Looking up on the table you will find $\Phi(0.8) = 0.7881$. So,

$$\sigma = \frac{4}{0.8} = 5 \qquad \sigma^2 = 25.$$

The variance is 25.

d. Let X be a binomial random variable with expected value 2.0 and variance 1.2. Find the parameters n and p for X.

$$\begin{aligned} E[X] &= n \cdot p = 2.0 \\ Var(X) &= n \cdot p \cdot (1-p) = 1.2 \\ &= 2.0 \cdot (1-p) \end{aligned}$$

So, $(1-p) = 0.6$, and $p = 0.4$. Solving the first equation, $n = 5$.

The parameters of X are $n = 5$ and $p = 0.4$.

3. (15 points)

For some constant c, the random variable X has the probability density function

$$f_X(x) = \begin{cases} c(1-x^2) & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c.

(b) Compute $E[X]$.

(c) Compute $Var(X)$.

(a).

$$\begin{aligned} 1 &= \int_{-1}^{1} c(1-x^2)\,dx \\ &= c\big(x - \frac{x^3}{3}\Big|_{x=-1}^{1}\big) \\ &= \frac{4}{3} \end{aligned}$$

So, $c = \frac{3}{4}$.

(b).

$$\begin{aligned} E[X] &= \frac{3}{4}\int_{-1}^{1} x(1-x^2)\,dx \\ &= \frac{3}{4}\big(\frac{x^2}{2} - \frac{x^4}{4}\Big|_{x=-1}^{1}\big) \\ &= 0 \end{aligned}$$

(c).

$$\begin{aligned} E[X^2] &= \frac{3}{4}\int_{-1}^{1} x^2(1-x^2)\,dx \\ &= \frac{3}{4}\big(\frac{x^3}{3} - \frac{x^5}{5}\Big|_{x=-1}^{1}\big) \\ &= \frac{1}{5} \end{aligned}$$

So,

$$Var(X) = E[X^2] - \big(E[X]\big)^2 = \frac{1}{5}.$$

4. (10 points)
Bippy's Bearing Works manufactures bearing shafts whose diameters are normally distributed with mean 1 and standard deviation 0.002 cm. Their specifications require the diameters to be in the range 1.000 ± 0.003 cm.

(a) What percentage of shafts are rejected?
The probability must be given to four decimal places.

(b) What is the maximum standard deviation that will permit no more than 1% of their shafts to be rejected?

(a). Let X denote the width of a bearing shaft. A shaft is rejected exactly when X lies outside the interval $[0.997, 1.003]$:

$$\begin{aligned}
1 - \mathbf{P}\{0.997 \le X \le 1.003\} &= 1 - \mathbf{P}\{\frac{0.997 - 1.000}{0.002} \le \frac{X - 1.000}{0.002} \le \frac{1.003 - 1.000}{0.002}\} \\
&= 1 - \mathbf{P}\{-\frac{3}{2} \le Z \le \frac{3}{2}\} \\
&= 1 - \Phi(\frac{3}{2}) + \Phi(-\frac{3}{2}) \\
&= 2\big(1 - \Phi(\frac{3}{2})\big) \\
&= 2(1 - 0.9332) = 2 \cdot 0.0668 = 0.1336.
\end{aligned}$$

(b). We want to choose σ so that

$$0.01 \le 2\big(1 - \Phi(\frac{0.003}{\sigma})\big)$$

Equivalently,

$$\Phi(\frac{0.003}{\sigma}) \le 0.995$$

Looking up on the table we must have

$$\frac{0.003}{\sigma} \ge 2.58$$

or $\sigma \le \frac{0.003}{2.58} = \frac{1}{860} \approx 0.0011$.

5. (10 points)
The number of hours of use of a lightbulb is exponentially distributed, with an expected lifetime of 1000 hours.

(a) Compute the probability that the light bulb is still working after 1000 hours.

(b) Compute the probability that a light bulb has at least another 500 hours of use given that it has already been in use 500 hours.

The parameter to the exponential random variable denoting the number of hours of life of a lightbulb is $\lambda = \frac{1}{1000}$. (Remember, expected lifetime is $\frac{1}{\lambda}$.)

(a). The probability the lightbulb is still working after 1000 hours is e^{-1}.

(b). The probability the lightbulb is still working after 500 hours is $e^{-0.5}$. (Recall that the exponential distribution is memoryless).

6. (10 points)
You are submitting a sealed bid to paint a house. If you have the lowest bid, then you will win the contract. You intend to pay Acme Painting 5 thousand dollars to do the work. If you believe the lowest competing bid (in thousands of dollars) of the other participating contractors will be uniformly distributed in the interval $(4, 9)$, how much should you bid to maximize your profit?

Let X denote your bid and Y denote the lowest competing bid. So, $X - 5$ is your expected profit if $X < Y$ and 0 otherwise. If you bid $4 \leq x \leq 9$, then your expected profit is

$$\begin{aligned} E[X] &= 0 \cdot \mathbf{P}\{x \geq Y\} + (x-5) \cdot \mathbf{P}\{x < Y\} \\ &= (x-5) \cdot (\frac{9-x}{5} \\ &= \frac{1}{5}\big(-x^2 + 14x - 45\big) \end{aligned}$$

The maximum of this downward opening parabola is where the derivative is zero:

$$-2x + 14 = 0$$

So, $x = 7$ thousand dollars maximizes the expected profit.