MATH 425

Midterm 1

Winter, 2009

Name:

- You have **50 minutes** to complete your work.
- Show all work and make it clear what your answers are.
- You are permitted one 3x5 notecard. Otherwise, books, notes, calculators and computers are not permitted on this exam.

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problem	points	score
1	15	
2	15	
3	10	
4	10	
5	20	
Total	70	

1. (5 points each)

a. How many arrangements of the letters REMEMBER ?

$$\frac{8!}{3! \cdot 2! \cdot 2!}$$

b. How many 5 digit numbers can be formed from the nine digits $\{1, 2, \ldots, 9\}$ if no digit can appear more than twice?

There are three distinct types of 5 digit numbers that can appear:

- All numbers distinct: $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$.
- One number occurring twice: $\binom{5}{2} \cdot 9 \cdot 8 \cdot 7 \cdot 6$. Choose a number a number to be repeated, choose two places out of five, then choose the remaining 3 numbers to be distinct.
- Two numbers occurring twice: $\binom{9}{2} \cdot \binom{5}{2! \cdot 2!} \cdot 7$. Choose the two numbers to be repeated (out of nine possible), choose the places, then choose the remaining number to be different.

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \ + \ \binom{5}{2} \cdot 9 \cdot 8 \cdot 7 \cdot 6 \ + \ \binom{9}{2} \cdot \frac{5!}{2! \cdot 2!} \cdot 7$$

A common mistake was to guess

$$9 \cdot 9 \cdot 8 \cdot 8 \cdot 7,$$

but this undercounts: if the choice for the first two spots are different, then there will be 9 choices available for the third spot. So, how many choices are available depends on the choices made earlier, and the Product Rule is not applicable.

c. How many 5 digit numbers can be formed from the nine digits $\{1, 2, \ldots, 9\}$ if no two consecutive digits can be the same?

The first spot can have any of 9 digits, but each succeeding spot cannot have the previous digit, so there are only 8 choices for spots 2 to 5:

$$9 \cdot 8^4$$

2. (2 points each) Consider three events A, B, C with

$$\mathbf{P}(A) = 0.4 \qquad \mathbf{P}(B) = 0.5 \qquad \mathbf{P}(C) = 0.7.$$

a. What is the probability of $\mathbf{P}(A \mid B)$ if A and B are independent?

$$\mathbf{P}(A \mid B) = \mathbf{P}\, A = 0.4.$$

b. What is the probability of $\mathbf{P}(A \mid B)$ if A and B are mutually exclusive?

$$\mathbf{P}(A \mid B) = 0 \quad \text{since } \mathbf{P}(A \cap B) = 0.$$

c. What is the probability of $\mathbf{P}(A \mid B)$ if $\mathbf{P}(A \cap B) = 0.2$?

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{2}{5}.$$

Alternatively, since $\mathbf{P}(A \cap B) = \mathbf{P}\, A \cdot \mathbf{P}(B)$, it follows that A and B are independent, so

$$\mathbf{P}(A \mid B) = \mathbf{P}(A) = 0.4.$$

d. What is smallest value possible value of $\mathbf{P}(B \cap C)$?

$$\mathbf{P}(B \cap C) \geq 0.2 \quad \text{since } \mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(B \cap C) \leq 1.$$

e. (2 pts) What is the largest possible value of $\mathbf{P}(B \cap C)$?

$$\mathbf{P}(B \cap C) \leq \mathbf{P}(B) = 0.5$$

f. (5 pts) What is $\mathbf{P}(A \cup B \cup C)$ if each of A, B and C are independent?

We have the following probabilities

$$\mathbf{P}(A \cap B) = 0.2 \quad \mathbf{P}(A \cap C) = 0.28 \quad \mathbf{P}(B \cap C) = 0.35 \quad \mathbf{P}(A \cap B \cap C) = 0.14$$

Use the Inclusion-Exclusion Rule:

$$\mathbf{P}(A \cup B \cup C) = 0.4 + 0.7 + 0.5 - 0.2 - 0.35 - 0.28 + 0.14 = 0.91$$

3. (5 points each)

a. A person chooses a letter at random from RESERVE, and independently one at random from VERTIGO. What is the probability that the same letter is chosen.

The probabilities for choosing the same letter, for each letter of RESERVE:

$$\mathbf{P}(R) = \frac{2}{7} \cdot \frac{1}{7} \quad \mathbf{P}(E) = \frac{3}{7} \cdot \frac{1}{7} \quad \mathbf{P}(S) = 0 \quad \mathbf{P}(V) = \frac{1}{7} \cdot \frac{1}{7}$$

So, the probability of choosing the same letter is

$$\mathbf{P}(R) + \mathbf{P}(E) + \mathbf{P}(S) + \mathbf{P}(V) = \frac{2}{49} + \frac{3}{49} + \frac{1}{49} = \frac{6}{49}$$

b. A forest preserve contains 20 elk, of which 5 are captured, tagged and released. A certain time later 4 of the elk are captured. What is the probability that 2 of the 4 have been tagged? (Assume that each elk is equally likely to be captured at any time.)

$$\frac{\binom{5}{2} \cdot \binom{15}{2}}{\binom{20}{4}} = \frac{70}{323}$$

4. (10 points) Urn A contains 3 white balls and 2 black balls. Urn B contains 1 white ball and 4 black balls. A ball is drawn at random from urn A and placed into urn B. Urn B is thoroughly mixed and a ball is drawn.

(a) Suppose a white ball is drawn from urn B. What is the probability that the ball transferred from urn A is white?

(b) Suppose a black ball is drawn from urn B. Now, what is the probability that the ball transferred from urn A is white?

Let H_W (H_B) be the hypothesis that the ball from urn A was white (black); let W (B) be the event that the ball drawn from B was white (black). Then

$$\mathbf{P}(H_W) = \frac{3}{5} \quad \mathbf{P}(H_B) = \frac{2}{5}$$
$$\mathbf{P}(W \mid H_W) = \frac{2}{6} \quad \mathbf{P}(W \mid H_B) = \frac{1}{6}.$$

By the Partition Rule,

$$\begin{aligned}\mathbf{P}(W) &= \mathbf{P}(W \mid H_W) \cdot \mathbf{P}(H_W) + \mathbf{P}(W \mid H_B) \cdot \mathbf{P}(H_B) \\ &= \frac{2}{6} \cdot \frac{3}{5} + \frac{1}{6} \cdot \frac{2}{5} = \frac{4}{15}\end{aligned}$$

(a).

$$\begin{aligned}\mathbf{P}(H_W \mid W) &= \mathbf{P}(W \mid H_W) \cdot \frac{\mathbf{P}(H_W)}{\mathbf{P}(W)} \\ &= \frac{2}{6} \cdot \frac{\frac{3}{5}}{\frac{4}{15}} = \frac{3}{4}\end{aligned}$$

(b). The probabilities we need are easily computed from those above:

$$\begin{aligned}\mathbf{P}(B) &= 1 - \mathbf{P}(W) = \frac{11}{15} \\ \mathbf{P}(B \mid H_W) &= 1 - \mathbf{P}(W \mid H_W) = \frac{2}{3}\end{aligned}$$

So,

$$\begin{aligned}\mathbf{P}(H_W \mid B) &= \mathbf{P}(B \mid H_W) \cdot \frac{\mathbf{P}(H_W)}{\mathbf{P}(B)} \\ &= \frac{2}{3} \cdot \frac{\frac{3}{5}}{\frac{11}{15}} = \frac{6}{11}\end{aligned}$$

5. (20 points) The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed genes, then the person will have brown eyes; and if one of them is a blue-eyed gene and the other a brown-eyed gene, then the person will have brown eyes. A newborn child independently receives one eye gene from each parent and the gene it receives from a parent is equally likely to be either of the two eye genes of that parent. Suppose that Smith's sister and mother have blue eyes, but Smith's father has brown eyes.

(a) What is the probability that Smith has blue eyes?

Suppose Smith's wife has blue eyes.

(b) What is the probability that their first child will have blue eyes?

(c) If their first child has blue eyes, what is the probability that their next child will also have blue eyes?

(d) If their first child has brown eyes, what is the probability that their next child will also have brown eyes?

Let B be the brown-eye gene and b the blue eye gene. Then, we know

- Smith's mother (bb), Smith's father (Bb or bB), Smith's wife (bb)

Let H_B be the hypothesis Smith carries B gene. Then,

$$\mathbf{P}(H_B) = \frac{1}{2},$$

since only Smith's father could pass along a B gene.

(a). Smith has blue eyes exactly when he has no B gene: H_B^c. So,

$$\mathbf{P}(\text{Smith is } bb) = \frac{1}{2}.$$

A common mistake is to believe Smith's father could be BB: that is, $\mathbf{P}(\text{father has } b) = \frac{2}{3}$. This leads to the probability $\frac{1}{3}$:

$$\begin{aligned} \mathbf{P}(\text{Smith is } bb) &= \mathbf{P}(\text{Smith } bb \,|\, \text{father has no } b) \cdot \mathbf{P}(\text{father has no } b) + \\ &\qquad \mathbf{P}(\text{Smith } bb \,|\, \text{father has } b) \cdot \mathbf{P}(\text{father has } b) \\ &= 0 + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}. \end{aligned}$$

(b). Smith is either Bb, bB or bb. Let C_b be Smith's child has blue eyes.

$$\begin{aligned}\mathbf{P}(C_b) &= \mathbf{P}(C_b \mid H_B) \cdot \mathbf{P}(H_B) + \mathbf{P}(C_b \mid H_B^c) \cdot \mathbf{P}(H_B^c) \\ &= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4}.\end{aligned}$$

(c). Let C_b be the event that the first child has blue eyes. We need to recompute H_B in the light of this new information.

$$\begin{aligned}\mathbf{P}(H_B \mid C_b) &= \frac{\mathbf{P}(C_b \mid H_B) \cdot \mathbf{P}(H_B)}{\mathbf{P}(C_b)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \\ \mathbf{P}(H_B^c \mid C_b) &= \frac{2}{3}.\end{aligned}$$

Let $\mathbf{P}_{C_b}(\cdot) = \mathbf{P}(\cdot \mid C_b)$ and D_b be the event that the second child has blue eyes. Then

$$\begin{aligned}\mathbf{P}_{C_b}(D_b) &= \mathbf{P}_{C_b}(D_b \mid H_B) \cdot \mathbf{P}_{C_b}(H_B) + \mathbf{P}_{C_b}(D_b \mid H_B^c) \cdot \mathbf{P}_{C_b}(H_B^c) \\ &= \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{5}{6}\end{aligned}$$

A common mistake is to assume that the events C_b and D_b are independent. They are NOT independent, but they are *conditionally independent* given the genetic type of the mother and father. So, while the following is false

$$\mathbf{P}(C_b \cap D_b) = \mathbf{P}(C_b) \cdot \mathbf{P}(D_b),$$

it is true that

$$\mathbf{P}(C_b \cap D_b \mid H_B) = \mathbf{P}(C_b \mid H_B) \cdot \mathbf{P}(D_b \mid H_B).$$

One way to see why this is the case is that once you know the first child has blue eyes, this changes the probability that Smith has a B gene:

$$\mathbf{P}(H_B \mid C_b) = \frac{1}{3} < \frac{1}{2} = \mathbf{P}(H_B),$$

which means the probability that both children have blue eyes is greater than what it would be if the events were independent:

$$\mathbf{P}(C_b) \cdot \mathbf{P}(D_b) = \frac{9}{16} < \mathbf{P}(C_b \cap D_b) = \frac{5}{6} \cdot \frac{3}{4} = \frac{15}{24}.$$

(d). Let C_B be the event that the first child has brown eyes. We need to recompute H_B in the light of this new information. But, Smith must now be Bb or bB, since the first child could only get a B gene from Smith.

$$\begin{aligned}\mathbf{P}(H_B \,|\, C_b) &= 1 \\ \mathbf{P}(H_B \,|\, C_b) &= 0.\end{aligned}$$

We also have that the probability that the first and second child receive B genes are conditionally independent of H_B (Smith's having the B gene). That is, if Smith has the B gene, there is a probability of $\frac{1}{2}$ he passes it on to his second child, regardless of the eye color of his first child. So,

$$\mathbf{P}(\text{second child brown-eyed} \,|\, \text{first child brown-eyed}) = \frac{1}{2}.$$