MATH 425
HOMEWORK 9
Winter, 2009

27a. The respective densities of X and Y are

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$
$$f_Y(y) = \begin{cases} e^{-y} & \text{if } 0 < x \\ 0 & \text{otherwise.} \end{cases}$$

The density of the sum $Z = X+Y$ can be found by convolution. The relevant formula is

$$f_Z(a) = \int_0^1 f_Y(a-x) f_X(x)\, dx$$

There are two cases (since $f_Y(a-x) = 0$ unless $a - x > 0$.) For $0 < a < 1$,

$$\begin{aligned} f_Z(a) &= \int_0^a e^{x-a}\, dx \\ &= 1 - e^{-a} \end{aligned}$$

For $a > 1$,

$$\begin{aligned} f_Z(a) &= \int_0^1 e^{x-a}\, dx \\ &= e^{1-a} - e^{-a} \end{aligned}$$

So, the density for $Z = X + Y$ is

$$f_Z(a) = \begin{cases} 1 - e^{-a} & \text{if } 0 < a < 1 \\ e^{1-a} - e^{-a} & \text{if } 1 < a \\ 0 & \text{otherwise.} \end{cases}$$

The cumulative distribution again depends on two cases. For $0 < a < 1$,

$$\begin{aligned} F_Z(a) &= \int_0^a 1 - e^{-t}\, dt \\ &= t + e^{-t}\Big|_{t=0}^{a} \\ &= a + e^{-a} - 1 \end{aligned}$$

For $a > 1$,

$$\begin{aligned} F_Z(a) &= \int_0^1 1 - e^{-t}\, dt + \int_1^a e^{1-t} - e^{-t}\, dx \\ &= t + e^{-t}\Big|_{t=0}^{1} + e^{-t} - e^{1-t}\Big|_{t=1}^{a} \\ &= 1 + e^{-a} - e^{1-a} \end{aligned}$$

The cumulative distribution function for $Z = X + Y$ is

$$F_Z(a) = \begin{cases} a + e^{-a} - 1 & \text{if } 0 < a < 1 \\ 1 + e^{-a} - e^{1-a} & \text{if } 1 < a \\ 0 & \text{otherwise.} \end{cases}$$

Note: It was not necessary to compute the density first (which is probably longer). You could also work with the joint density function:

$$f_{X,Y}(x, y) = e^{-y} \qquad 0 < x < 1 \text{ and } 0 < y.$$

Then compute

$$\begin{aligned} \mathbf{P}\{Z \le a\} &= \mathbf{P}\{X + Y \le a\} \\ &= \mathbf{P}\{Y \le a - X\} \end{aligned}$$

There are two cases: for $0 < a < 1$,

$$\mathbf{P}\{Z \le a\} = \int_0^a \int_0^{a-x} e^{-y}\, dy\, dx = a + e^{-a} - 1;$$

and for $1 < a$,

$$\mathbf{P}\{Z \le a\} = \int_0^1 \int_0^{a-x} e^{-y}\, dy\, dx = 1 + e^{-a} - e^{1-a};$$

27b. We compute the distribution of $Z = X/Y$ directly using the joint density $f_{X,Y}(x,y)$ as in the Note above. Let $a > 0$,

$$\begin{aligned}
\mathbf{P}\{Z \le a\} &= \mathbf{P}\{X/Y \le a\} \\
&= \mathbf{P}\{X \le aY\} \\
&= \int_0^\infty \int_0^1 f_{X,Y}(x,y)\, dx\, dy \\
&= \int_0^{1/a} \int_0^{ay} e^{-y}\, dy + \int_{1/a}^\infty \int_0^1 e^{-y}\, dx\, dy \\
&= a\big(1 - e^{-1/a}\big).
\end{aligned}$$

The cumulative distribution for $Z = X/Y$ is

$$F_Z(a) = \begin{cases} a\big(1 - e^{-1/a}\big) & \text{if } 0 < a \\ 0 & \text{otherwise.} \end{cases}$$

Note. An alternative method (which is quicker) is

$$\begin{aligned}
\mathbf{P}\{Z \le a\} &= \mathbf{P}\{X/Y \le a\} \\
&= \mathbf{P}\{X/a \le Y\} \\
&= \int_0^1 \int_{x/a}^\infty e^{-y}\, dy = a\big(1 - e^{-1/a}\big).
\end{aligned}$$

31a. The number of airplane accidents in one month can be modeled by a Poisson random variable, where $\lambda = 2.2$. The sum of n independent Poisson variables with mean λ is a Poisson random variable with mean $n \cdot \lambda$ (see Example 6.3e in Ross). In Problem **31b** and **31c** I will assume the accidents in any given month are independent of those in any other month.

Let X count the number of accidents next month.

$$\begin{aligned}
\mathbf{P}\{X > 2\} &= 1 - \mathbf{P}\{X \le 2\} = 1 - \mathbf{P}\{X = 0\} - \mathbf{P}\{X = 1\} - \mathbf{P}\{X = 1\} \\
&= 1 - e^{-2.2} 2.2 e^{-2.2} - \frac{2.2^2}{2} e^{-2.2} \\
&\approx 0.3773
\end{aligned}$$

31b. Let Y count the number of accidents the next two months, so $\lambda = 2 \cdot 2.2 = 4.4$.

$$\begin{aligned}
\mathbf{P}\{Y > 4\} &= 1 - \mathbf{P}\{Y \le 4\} = 1 - \sum_{i=0}^{4} \mathbf{P}\{Y = i\} \\
&= 1 - e^{-4.4} - 4.4e^{-4.4} - e^{-4.4}\frac{4.4^2}{2} - e^{-4.4}\frac{4.4^3}{3!} - e^{-4.4}\frac{4.4^4}{4!} \\
&\approx 0.4488
\end{aligned}$$

31c. Let Z count the number of accidents the next three months, so $\lambda = 3 \cdot 2.2 = 6.6$.

$$\begin{aligned}
\mathbf{P}\{Z > 5\} &= 1 - \mathbf{P}\{Z \le 5\} = 1 - \sum_{i=0}^{5} \mathbf{P}\{Z = i\} \\
&= 1 - \sum_{i=0}^{5} e^{-6.6}\frac{6.6^i}{i!} \\
&\approx 0.6453
\end{aligned}$$

33a. Jill's bowling score is normally distributed with $\mu_a = 170$ and $\sigma_a = 20$ and Jack's bowling score is normally distributed with $\mu_b = 160$ and $\sigma_b = 15$. Let X be Jill's score and Y be Jack's score, so we want $\mathbf{P}\{Y - X > 0\}$. Because the scores are independent, the random variable $Y - X$ is also normally distributed with

$$\mu = \mu_b - \mu_a = 160 - 170 = -10 \qquad \sigma^2 = \sigma_a^2 + \sigma_b^2 = 400 + 225 = 625.$$

This is by Section 5.4, $-X$ is normally distributed with mean $-\mu_a$ and variance $(-1)^2\sigma_a^2$; $Y + (-X)$ is normally distributed by Proposition 6.3.2.

Use continuity correction to compute the probability (where Z is standard):

$$
\begin{aligned}
\mathbf{P}\{Y - X > 0\} &= \mathbf{P}\{Y - X > 0.5\} \\
&= \mathbf{P}\{\frac{Y - X - \mu}{\sigma} > \frac{0.5 - \mu}{\sigma}\} \\
&= \mathbf{P}\{Z > \frac{10.5}{\sqrt{625}}\} \\
&= \mathbf{P}\{Z > 0.42\} \\
&= 1 - \mathbf{P}\{Z \leq 0.42\} \\
&\approx 1 - 0.6628 \\
&= 0.3372.
\end{aligned}
$$

33b. The total of their scores, $X + Y$, is normally distributed with

$$\mu = \mu_b + \mu_a = 160 + 170 = 330 \qquad \sigma^2 = \sigma_a^2 + \sigma_b^2 = 400 + 225 = 625.$$

Applying continuity correction

$$
\begin{aligned}
\mathbf{P}\{X + Y > 350\} &= \mathbf{P}\{X + Y > 350.5\} \\
&= \mathbf{P}\{\frac{X + Y - \mu}{\sigma} > \frac{350.5 - \mu}{\sigma}\} \\
&= \mathbf{P}\{Z > \frac{20.5}{\sqrt{625}}\} \\
&= \mathbf{P}\{Z > 0.42\} \\
&= 1 - \mathbf{P}\{Z \leq 0.82\} \\
&\approx 1 - 0.7939 \\
&= 0.2061.
\end{aligned}
$$

34a. Let X count the number of men in the 200 sampled never eat breakfast and Y count the number of women in the 200 sampled never eat breakfast. X is a binomial random variable with parameters $n = 200, p = 0.252$ and Y is a binomial random variable with parameters $n = 200, p = 0.236$. We will approximate these random variables by normal random variables X_N and Y_N whose parameters are

$$
\begin{array}{lll}
X_N: & \mu_X = 200 \cdot 0.252 = 50.4 & \sigma_X^2 = 200 \cdot 0.252 \cdot (1 - 0.252) \approx 36.7 \\
Y_N: & \mu_Y = 200 \cdot 0.236 = 47.2 & \sigma_Y^2 = 200 \cdot 0.236 \cdot (1 - 0.236) \approx 36.06.
\end{array}
$$

The sum $Z = X_N + Y_N$ is a normal random variable with parameters which are the sum of those for X_N and Y_N (see Proposition 6.3.2 in Ross):

$$Z_N : \quad \mu_Z = 50.4 + 47.2 = 97.6 \qquad \sigma_Z^2 = 36.7 + 36.06 = 73.76.$$

The probability that at least 110 out of 400 people sampled never eat breakfast is

$$\begin{aligned} \mathbf{P}\{X + Y \geq 110\} &\approx \mathbf{P}\{X_N + Y_N \geq 109.5\} = \mathbf{P}\{Z_N \geq 109.5\} \\ &= \mathbf{P}\{\frac{Z_N - 97.6}{\sqrt{73.76}} \geq \frac{109.5 - 97.6}{\sqrt{73.76}}\} \\ &= 1 - \Phi(1.39) = 1 - 0.9117 = 0.0823. \end{aligned}$$

Note. Ross interpolates to obtain their value. A closer estimate (to three places) for the table look-up is

$$1 - \Phi(1.386) \approx 1 - 0.9171 = 0.0829.$$

34b. We will use the normal approximations from **34a**. Remember for any normal random variable N with parameters μ, σ^2, the random variable $aN+b$ is normally distributed with parameters $a\mu + b, a^2\sigma^2$. (See Ross, Section 5.4, pages 219-20.)

The random variable $-Y_N$ is normally distributed with $-\mu_Y, \sigma_Y^2$ and the sum $X_N - Y_N$ is normally distributed with $\mu_X - \mu_Y = 3.2, \sigma_X^2 + \sigma_X^2 = 73.76$.

$$\begin{aligned} \mathbf{P}\{X \leq Y\} &= \mathbf{P}\{X - Y \leq 0\} \approx \mathbf{P}\{X_N - Y_N \leq 0.5\} \\ &= \mathbf{P}\{\frac{(X_N - Y_N) - 3.2}{\sqrt{73.76}} \leq \frac{0.5 - 3.2}{\sqrt{73.76}} \\ &\approx \Phi(-0.31) = 1 - \Phi(0.31) = 1 - 0.6217 = 0.3783 \end{aligned}$$

Note. Ross interpolates to obtain their value. A closer estimate (to three places) for the table look-up is

$$1 - \Phi(0.314) \approx 1 - 0.6234 = 0.3766.$$

41a. The condition mass function $p_{X|Y}$ is

$$\begin{aligned} p_{X|Y}(1|1) &= \frac{1/8}{1/4} = \frac{1}{2} \qquad & p_{X|Y}(2|1) &= \frac{1/8}{1/4} = \frac{1}{2} \\ p_{X|Y}(1|2) &= \frac{1/4}{3/4} = \frac{1}{3} \qquad & p_{X|Y}(2|3) &= \frac{1/2}{3/4} = \frac{2}{3} \end{aligned}$$

41b. If X is independent of Y we must have $p_X(i) = pX|Y(i|j)$ for any i and j. Since the conditional probabilities of X given Y vary when $Y = 2$, X and Y are NOT independent.

Here is a computation of p_X:

$$p_X(1) = \frac{3}{8} \qquad p_X(2) = \frac{5}{8}.$$

This is a verification that X and Y are NOT independent, since $p_X(1) = \frac{3}{8}$ but $p_{X|Y}(1|1) = \frac{1}{2}$.

41c. (i). Since $XY \leq 3$ is true, except when $X = 2, Y = 2$,

$$\mathbf{P}\{XY \leq 3\} \;=\; 1 - p_{X,Y}(2,2) = 1 - \frac{1}{2} = \frac{1}{2}.$$

(ii). $X + Y > 2$ when $X \geq 2$ or $Y \geq 2$; however, this is always the case, except when $X = 1, Y = 1$,

$$\mathbf{P}\{X + Y > 2\} \;=\; 1 - p_{X,Y}(1,1) = 1 - \frac{1}{8} = \frac{7}{8}.$$

(iii). $X/Y > 1$ when $X > Y$,

$$\mathbf{P}\{X/Y > 1\} \;=\; p_{X,Y}(2,1) = \frac{1}{8}.$$

43. We are given the joint density for X and Y

$$f_{X,Y}(x,y) = c(x^2 - y^2)e^{-x} \qquad 0 \leq x < \infty, -x \leq y \leq x.$$

We first compute the marginal density $f_X(x)$ and then the conditional density function $f_{X|X}(y|x)$. For the marginal density

$$\begin{aligned} f_X(x) &= \int_{-\infty} \infty f_{X,Y}(x,y)\,dy \\ &= \int_{-x} xc(x^2 - y^2)e^{-x} \\ &= c\frac{4}{3}x^3e^{-x}. \end{aligned}$$

The conditional density function of Y given $X = x$ is 0 unless $-x \leq y \leq x$,

$$\begin{aligned} f_{X|Y}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{c(x^2 - y^2)e^{-x}}{c\frac{4}{3}x^3e^{-x}} \\ &= \frac{3}{4}\frac{x^2 - y^2}{x^3}. \end{aligned}$$

The conditional probability distribution of Y given $X = x$ when $-x \leq a \leq x$ is

$$\begin{aligned} F_{X|Y}(a|x) &= \int_{-\infty}^{a} f_{X|Y}(y|x)\,dy \\ &= \int_{-x}^{a} \frac{3}{4}\frac{x^2 - y^2}{x^3}\,dy \\ &= \frac{1}{2} + \frac{3a}{4x} - \frac{a^3}{4x^3}. \end{aligned}$$

The conditional probability distribution of Y given $X = x$ is

$$F_{X|Y}(a|x) = \begin{cases} 0 & \text{if } a < -x \\ \frac{1}{2} + \frac{3a}{4x} - \frac{a^3}{4x^3} & \text{if } -x \leq a \leq x \\ 1 & \text{otherwise.} \end{cases}$$

54a. The joint density function of X and Y was given as

$$f(x,y) = \frac{1}{x^2}y^2 \qquad x, y \geq 1.$$

The transformation from the xy-plane to the uv-plane is

$$u = xy \qquad v = \frac{x}{y}$$

and is one-to-one, and so has an inverse

$$x = \sqrt{u \cdot v} \qquad y = \sqrt{\frac{u}{v}}.$$

The Jacobian determinant is

$$J(x,y) = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -2\frac{x}{y} = -2v.$$

So, the joint density for U and V is

$$f_{U,V}(u,v) = f_{X,Y}\big(\sqrt{uv}, \sqrt{\frac{u}{v}}\big) \cdot \big| -\frac{1}{2v}\big| = \begin{cases} \frac{1}{2u^2v} & \text{if } 1 \le \sqrt{uv}, \frac{u}{v} \\ 0 & \text{otherwise.} \end{cases}$$

Now, $0 < u, v$ and

$$1 \le \sqrt{uv}, 1 \le \frac{u}{v} \implies 0 < v \le u, 0 < \frac{1}{v} \le u.$$

These conditions are only consistent when $u \in (1, \infty)$, and in this case $\frac{1}{u} \le v \le u$. So, we have

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{2u^2v} & \text{if } 1 \le u, \frac{1}{u} \le v \le u \\ 0 & \text{otherwise.} \end{cases}$$

54b. The marginal for U is obtained by integrating the joint density $f_{U,V}(u,v)$ over v

$$f_U(u) = \int_{\frac{1}{u}}^{u} \frac{1}{2u^2v}\, dv = \frac{\ln u}{u^2} \qquad u \ge 1.$$

So,

$$f_U(u) = \begin{cases} \frac{\ln u}{u^2} & \text{if } u \ge 1 \\ 0 & \text{otherwise.} \end{cases}$$

The marginal for V is obtained by integrating the joint density $f_{U,V}(u,v)$ over u. There are two possibilities, when $v \in (0,1)$:

$$f_V(v) = \int_{\frac{1}{v}}^{\infty} \frac{1}{2u^2v}\, du = \frac{1}{2}$$

and $v \in [1, \infty)$:

$$f_V(v) = \int_{v}^{\infty} \frac{1}{2u^2v}\, du = \frac{1}{2v^2}$$

So,

$$f_V(v) = \begin{cases} \frac{1}{2} & \text{if } 0 < v < 1 \\ \frac{1}{2v^2} & \text{if } 1 \le v \\ 0 & \text{otherwise.} \end{cases}$$

57. X_1 and X_2 are independent exponential random variables with parameter λ, so their joint density is

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \lambda^2 e^{-\lambda(x_1+x_2)} 0 & \text{otherwise.} \end{cases}$$

The transformation from the x_1x_2-plane to the y_1y_2-plane is

$$y_1 = x_1 + x_2 \qquad y_2 = e^{x_1}$$

and is one-to-one, and so has an inverse

$$x_1 = \ln y_2 \qquad x_2 = y_1 - \ln y_2.$$

The Jacobian determinant is

$$J(x_1,x_2) = \begin{vmatrix} 1 & 1 \\ e^{x_1} & 0 \end{vmatrix} = -e^{x_1} = -y_1.$$

So, the joint density for Y_1 and Y_2 is

$$f_{Y_1.Y_2}(y_1,y_2) = f_{X_1,X_2}\big(\ln y_2, y_1 - \ln y_2\big)\cdot\big|-y_1\big| = \begin{cases} \lambda^2 e^{-\lambda y_1} & \text{if } 0 \le \ln y_2, 0 \le y_1 - \ln y_2 \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$0 \le \ln y_2, 0 \le y_1 - \ln y_2 \implies 1 \le y_2, \ln y_2 \le y_1.$$

So,

$$f_{Y_1.Y_2}(y_1,y_2) = \begin{cases} \lambda^2 e^{-\lambda y_1} & \text{if } 1 \le y_2, \ln y_2 \le y_1. \\ 0 & \text{otherwise.} \end{cases}$$

56c. X and Y are independent exponential random variables with parameter $\lambda = 1$. Their individual densities are

$$f_X(x) = \begin{cases} e^{-x} \text{ if } x \ge 0 \\ 0 & \text{otherwise} \end{cases} \qquad f_U(y) = \begin{cases} e^{-y} \text{ if } y \ge 0 \\ 0 & \text{otherwise} \end{cases},$$

so their joint density is

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = e^{-(x+y)} \qquad x,y \ge 0.$$

Consider the random variables

$$U = X + Y \qquad V = \frac{X}{X+Y}.$$

The transformation

$$u = x + y \qquad v = \frac{x}{x+y}$$

has an inverse transformation

$$x = uv \qquad y = u - uv.$$

Compute the Jacobian for the $(uv \Rightarrow xy)$ transformation:

$$J(u,v) = \begin{vmatrix} v & 1-v \\ u & -u \end{vmatrix} = -uv - u(1-v) = -u$$

Compute the joint density:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(uv, u-uv) \cdot \big|J(u,v)\big| \\ &= ue^{-(uv+u-uv)} \\ &= ue^{-u} \end{aligned}$$

where u and v must satisfy

$$uv, u - uv \geq 0, \qquad \text{equivalently } 0 \leq uv \leq u;$$

Thus, the joint density of U and V

$$f_{U,V}(u,v) = \begin{cases} ue^{-u} & \text{if } u \geq 0,\ 0 \leq v \leq 1. \end{cases}$$

This is not part of the problem, but note that U and V are independent, V is uniformly distributed in $[0,1]$ and U is a gamma random variable with parameters $(\alpha = 2, \lambda = a)$. This makes sense: X and Y are the waiting times for the first success of independent events with the same expected waiting time. Then $X + Y$ is the waiting time for two events – gamma distributed with $\alpha = 2$, and the ratio $\frac{X}{X+Y}$ could be any value in $[0,1]$.