MATH 425
HOMEWORK 8
Winter, 2009

8a. We are given the joint probability density function of X and Y:

$$f(x,y) = \begin{cases} c(y^2 - x^2)e^{-y} & \text{if } 0 < y < \infty, -y \le x \le y, \\ 0 & \text{otherwise .} \end{cases}$$

Since f is a joint density function, choose c to satisfy

$$\begin{aligned} 1 &= \int_0^\infty \int_{-y}^{y} c(y^2 - x^2)e^{-y}\,dx\,dy \\ &= \int_0^\infty ce^-y\Big(\int_{-y}^{y}(y^2 - x^2)\,dx\Big)\,dy \\ &= c\int_0^\infty \frac{4}{3}y^3e^{-y}\,dy \\ &= 8c\int_0^\infty \frac{e^{-y}y^3}{3!}\,dy \\ &= 8c. \end{aligned}$$

So, $c = \frac{1}{8}$. Note that the function inside the last integral

$$\frac{e^{-y}y^3}{3!}$$

is the density of a gamma distribution with parameters $(k = 3, \lambda = 1)$, so the integral is 1.

8b. The marginal density of X is

$$\begin{aligned} f_X(a) &= \int_{-\infty}^{\infty} f(a,y)\,dy \\ &= \int_{|a|}^{\infty} \frac{1}{8}(y^2 - a^2)e^{-y}\,dy \\ &= \frac{1}{4}\big(|a| + 1\big)e^{-|a|} \end{aligned}$$

The second equality is because $f(a,y) = 0$ when $y < 0$ or when $a \notin (-y,y)$, so that $y > |a|$.

The marginal density of Y is

$$\begin{aligned} f_Y(b) &= \int_{-\infty}^{\infty} f(x,b)\,dx \\ &= \int_{-b}^{b} \frac{1}{8}(b^2 - x^2)e^{-b}\,dx \\ &= \frac{1}{6}b^3 e^{-b}. \end{aligned}$$

8c. The expectation of X can be computed from the marginal density from (8b)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x\frac{1}{4}\big(|x|+1\big)e^{-|x|}\,dx \\ &= \int_{-\infty}^{0} x\frac{1}{4}\big(|x|+1\big)e^{-|x|}\,dx + \int_{0}^{\infty} x\frac{1}{4}\big(|x|+1\big)e^{-|x|}\,dx \\ &= -\int_{0}^{\infty} x\frac{1}{4}\big(|x|+1\big)e^{-|x|}\,dx + \int_{0}^{\infty} x\frac{1}{4}\big(|x|+1\big)e^{-|x|}\,dx \\ &= 0. \end{aligned}$$

The third equality is by substituting $u = -x$ in the first integral.

9a. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{6}{7}\big(x^2 + \frac{xy}{2}\big) & \text{if } 0 < y < 2, 0 < x < 1, \\ 0 & \text{otherwise .} \end{cases}$$

This is really a density function: (a) $f(x,y) \geq 0$ for all x and y, and (b)

$$\int_0^2 \int_0^1 \frac{6}{7}\big(x^2 + \frac{xy}{2}\big)\,dx\,dy = \int_0^2 \Big(\frac{2}{7} + \frac{3y}{14}\Big)\,dy = 1.$$

9b. The density function for X is

$$\begin{aligned} f_X(a) &= \int_0^2 \frac{6}{7}\big(a^2 + \frac{ay}{2}\big)\,dy \\ &= \frac{12}{7}a^2 + \frac{6}{7}a = \frac{6}{7}(2a^2 + a) \end{aligned}$$

when $0 < a < 1$ and $f_X(a) = 0$ otherwise.

9c. Compute.

$$\begin{aligned}
\mathbf{P}\{X > Y\} &= \int_0^1 \int_0^x \frac{6}{7}\big(x^2 + \frac{xy}{2}\big)\, dy\, dx \\
&= \int_0^1 \Big(\frac{6}{7}x^2 + \frac{3}{14}x^3\Big)\, dx \\
&= \frac{15}{56}.
\end{aligned}$$

Alternatively, we can compute the probability by reversing the order of integration. To do this note that X can be at most one, so we need only integrate y up to 1.

$$\begin{aligned}
\mathbf{P}\{X > Y\} &= \int_0^1 \int_y^1 \frac{6}{7}\big(x^2 + \frac{xy}{2}\big)\, dx\, dy \\
&= \int_0^1 \Big(\frac{2}{7}(1 - y^3) + \frac{3}{14}(y - y^3)\Big)\, dy \\
&= \frac{15}{56}.
\end{aligned}$$

9d. By the definition of conditional probability:

$$\mathbf{P}(Y > \frac{1}{2} \,|\, X < \frac{1}{2}) = \frac{\mathbf{P}\{X < \frac{1}{2}, Y > \frac{1}{2}\}}{\mathbf{P}\{X < \frac{1}{2}\}}$$

Compute each probability.

$$\begin{aligned}
\mathbf{P}\{X < \frac{1}{2}, Y > \frac{1}{2}\} &= \int_{\frac{1}{2}}^2 \int_0^{\frac{1}{2}} \frac{6}{7}\big(x^2 + \frac{xy}{2}\big)\, dx\, dy \\
&= \int_{\frac{1}{2}}^2 \Big(\frac{1}{28} + \frac{3}{56}y\Big)\, dy \\
&= \frac{69}{448}
\end{aligned}$$

Use the density for X from (9b):

$$\mathbf{P}\{X < \frac{1}{2}\} = \int_0^{\frac{1}{2}} \frac{6}{7}(2x^2 + x)\, dx = \frac{5}{28}.$$

So, plugging back into the first equation

$$\mathbf{P}(Y > \frac{1}{2} \,|\, X < \frac{1}{2}) \;=\; \frac{69/448}{5/28} = \frac{69}{80} = 0.8625.$$

9e. Compute using the density from 9b.

$$E[X] \;=\; \int_0^1 x\frac{6}{7}(2x^2 + x)\, dx = \frac{5}{7}$$

9f. We will need the density function for Y.

$$\begin{aligned} f_Y(b) &= \int_0^1 \frac{6}{7}\big(x^2 + \frac{xb}{2}\big)\, dx \\ &= \frac{2}{7} + \frac{3}{14}b \end{aligned}$$

when $0 < b < 2$ and $f_Y(b) = 0$ otherwise.
Compute.

$$E[Y] \;=\; \int_0^2 y\big(\frac{2}{7} + \frac{3}{14}y\big)\, dy = \frac{8}{7}$$

10a. Compute.

$$\begin{aligned} \mathbf{P}\{X < Y\} &= \int_0^\infty \int_0^y e^{-(x+y)}\, dx\, dy \\ &= \int_0^\infty e^{-y} - e^{-2y}\, dy \\ &= 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Alternatively,

$$\begin{aligned} \mathbf{P}\{X < Y\} &= \int_0^\infty \int_x^\infty e^{-(x+y)}\, dy\, dx \\ &= \int_0^\infty e^{-2x}\, dx \\ &= \frac{1}{2}. \end{aligned}$$

10b. When $a < 0$, $\mathbf{P}\{X < a\} = 0$. When $a \geq 0$,

$$\begin{aligned} \mathbf{P}\{X < a\} &= \int_0^a \int_0^\infty e^{-(x+y)}\, dy \\ &= \int_0^a e^{-x}\, dx \\ &= 1 - e^{-a} \end{aligned}$$

Alternatively, notice that X and Y are independent and have the same distribution:

$$e^{-(x+y)} = e^{-x} e^{-y}.$$

which the exponential distribution with $\lambda = 1$. The cumulative distribution for X is then

$$\mathbf{P}\{X < a\} = 1 - e^{-a}$$

12. We assume the men and women are equally likely to enter the drugstore. Let M, W be random variables giving the number of men, women entering the store during the hour. From Example 6.2b, M and W are independent and are Poisson distributed with parameter $\lambda = 10 \cdot \frac{1}{2} = 5$. We want to compute

$$\mathbf{P}(M \leq 3 \,|\, W = 10) = \frac{\mathbf{P}\{M \leq 3, W = 10\}}{\mathbf{P}\{W = 10\}} = \mathbf{P}\{M \leq 3\}.$$

The last equality is because M and W are independent.

So, the probability that no more than 3 men enter the store is

$$\begin{aligned} \mathbf{P}\{M = 3\} &= e^{-5} + e^{-5} 5 + e^{-5} \frac{5^2}{2!} + e^{-5} \frac{5^3}{3!} \\ &= \frac{118}{3} e^{-5} \\ &\approx 0.265 \end{aligned}$$

13a. Let X be the time the man arrives and Y the time the woman arrives (both in fractions of 1 hour). We want the probability $\mathbf{P}\{|X - Y| \leq \frac{1}{12}\}$.

The probability densities for each variable are

$$f_X(t) = \begin{cases} 2 & \text{if } \frac{1}{4} \le t \le \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The probability is given by

$$\begin{aligned} \mathbf{P}\{|X - Y| \le \frac{1}{12}\} &= \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{t-\frac{1}{12}}^{t+\frac{1}{12}} 2\, ds\, dt \\ &= \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{3}\, dt \\ &= \frac{1}{6}. \end{aligned}$$

13b. We want to compute the probability $\mathbf{P}\{X < Y\}$.

$$\begin{aligned} \mathbf{P}\{X < Y\} &= \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{t}^{1} 2\, ds\, dt \\ &= \int_{\frac{1}{4}}^{\frac{3}{4}} 2 - 2t\, dt \\ &= \frac{1}{2}. \end{aligned}$$

14. Let X be the location of the ambulance (at the time of the accident) and Y the location of the accident. These random variables are independent and are uniformly distributed on the interval $[0, L]$ (the endpoints are irrelevant). Their density (which is the same) is given by

$$f(x) = \begin{cases} \frac{1}{L} & \text{if } 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

We need to determine the probabilities $\mathbf{P}\{|X - Y| \le a\}$ for $0 \le a \le L$. There are three regions: $\{0 \le X \le a\}$, $\{a \le X \le L-a\}$ and $\{L-a \le X \le L\}$

$$\begin{aligned}
\int_0^a \int_0^{x+a} \frac{1}{L^2}\,dx\,dy &= \frac{a^2}{L^2} \\
\int_a^{L-a} \int_{y-a}^{y+a} \frac{1}{L^2}\,dx\,dy &= \frac{2a}{L} - \frac{4a^2}{L^2} \\
\int_a^L \int_{x-a}^{L} \frac{1}{L^2}\,dx\,dy &= \frac{a^2}{L^2} \\
\mathbf{P}\{|X-Y| \le a\} &= \frac{a^2}{L^2} + \frac{2a}{L} - \frac{4a^2}{L^2} + \frac{a^2}{L^2} \\
&= \frac{2aL - a^2}{L^2}.
\end{aligned}$$

18. X and Y have similar densities, which are uniform over an interval of length $\frac{L}{2}$:

$$f_X(x) = \begin{cases} \frac{2}{L} & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and f_Y is the same except $f_Y(y) = \frac{2}{L}$ for $\frac{L}{2} \le y \le 1$.

There are two cases to compute $\mathbf{P}\{Y - X > \frac{L}{3}\}$, when $X < \frac{L}{6}$ (so Y can take any value) and when $\frac{L}{6} < X < \frac{L}{2}$ (so Y must be at least $X + \frac{L}{3}$):

$$\begin{aligned}
\int_0^{\frac{L}{6}} \int_{\frac{L}{2}}^{L} \frac{4}{L^2}\,dy\,dx &= \frac{1}{3} \\
\int_{\frac{L}{6}}^{\frac{L}{2}} \int_{x+\frac{L}{3}}^{\frac{L}{2}} \frac{4}{L^2}\,dy\,dx &= \frac{4}{9} \\
\mathbf{P}\{Y - X > \frac{L}{3}\} &= \frac{1}{3} + \frac{4}{9} = \frac{7}{9}.
\end{aligned}$$

19a. The joint density is given as

$$f(x,y) = \begin{cases} \frac{1}{x} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The marginal density of Y when $0 < y < 1$ is

$$\begin{aligned} f_Y(y) &= \int_y^1 \frac{1}{x}\,dx \\ &= \ln 1 - \ln y = -\ln y \end{aligned}$$

So,

$$f_Y(y) = \begin{cases} -\ln y & \text{if } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

19b. The marginal density of X when $0 < x < 1$ is

$$\begin{aligned} f_X(x) &= \int_0^x \frac{1}{x}\,dy \\ &= 1 \end{aligned}$$

So,

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

19c. The expectation of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}.$$

19d. The expectation of Y is

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\,dy \\ &= \int_0^1 -y \ln y\,dy \qquad \text{(by parts: } u = \ln y,\ dv = -y) \\ &= \left[-\frac{y^2 \ln y}{2}\Big|_{y=0}^{1} \right] + \int_0^1 \frac{y}{2}\,dy \\ &= 0 + \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

For the third equality, use l'Hôspital:

$$\begin{aligned}\lim_{y\to 0^+} -y^2 \ln y &= \lim_{y\to 0^+} \frac{-\ln y}{1/y^2} \\ &= \lim_{y\to 0^+} \frac{1/y}{2/y^3} \\ &= \lim_{y\to 0^+} \frac{y^2}{2} = 0.\end{aligned}$$

20a. The joint density of X and Y was is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)} & \text{if } 0 < x, y \\ 0 & \text{otherwise.} \end{cases}$$

X and Y are independent. The quickest way to see this is that $f(x,y)$ is the product of a function of x alone and a function of y alone: when $0 < x, y$

$$f(x,y) = \left(xe^{-x}\right) \cdot \left(e^{-y}\right).$$

When $x < 0$ or $y < 0$, the equation is still true. So, independence follows by Proposition 2.1.

20b. The joint density of X and Y was is given by

$$f(x,y) = \begin{cases} 2 & \text{if } 0 < x < y, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

X and Y are not independent. To see this compute the marginal density functions when $0 < x, y < 1$:

$$\begin{aligned} f_X(x) &= \int_x^1 2\,dy = 2(1-x) \\ f_Y(y) &= \int_0^y 2\,dy = 2y. \end{aligned}$$

However, for $0 < x < y < 1$

$$f(x,y) = 2 \neq 2(1-x) \cdot 2y = f_X(x) \cdot f_Y(y).$$

23a. The joint density of X and Y is

$$f(x,y) = \begin{cases} 12xy(1-x) & \text{if } 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

X and Y are independent. The quickest way to see this is that $f(x,y)$ is the product of a function of x alone and a function of y alone: when $0 < x, y < 1$

$$f(x,y) = \big(12x(1-x)\big) \cdot \big(y\big),$$

When one of x or y is not in the interval $(0,1)$, the equation is still true. So, independence follows by Proposition 2.1.

Since we will need the marginal densities, we can verify this by computing these. When $0 < x, y < 1$,

$$\begin{aligned} f_X(x) &= \int_0^1 12xy(1-x)\,dy = 6x(1-x)y^2\Big|_{y=0}^1 = 6x(1-x) \\ f_Y(y) &= \int_0^1 12xy(1-x)\,dx = \big(6x^2y - 4x^3y\big)\Big|_{x=0}^1 = 2y. \end{aligned}$$

both marginal densities are 0 otherwise.

Note that when $0 < x, y < 1$,

$$f(x,y) = 12xy(1-x) = f_X(x) \cdot f_Y(y),$$

verifying independence.

23b. Compute the expectation of X.

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\ &= \int_0^1 6x^2(1-x)\,dx = \frac{1}{2}. \end{aligned}$$

23c. Compute the expectation of Y.

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\, dx \\ &= \int_0^1 2y^2\, dy = \frac{2}{3}. \end{aligned}$$

23d. Compute the variance of X.

$$\begin{aligned} E[X] &= \frac{1}{2} \\ E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \\ &= \int_0^1 6x^3(1-x)\, dx = \frac{3}{10} \\ Var(X) &= E[X^2] - \big(E[X]\big)^2 \\ &= \frac{3}{10} - \frac{1}{4} = \frac{1}{20}. \end{aligned}$$

23e. Compute the variance of Y.

$$\begin{aligned} E[Y] &= \frac{2}{3} \\ E[Y^2] &= \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy \\ &= \int_0^1 2y^3\, dx = \frac{1}{2} \\ Var(X) &= E[X^2] - \big(E[X]\big)^2 \\ &= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}. \end{aligned}$$