MATH 425
HOMEWORK 7
Winter, 2009

23a. We want the probability that a 6 will appear in between 150 and 200 out of 1000 throws. This is a binomial distribution with $n = 1000$ and $p = \frac{1}{6}$. So,

$$\mu = np = \frac{1000}{6} \qquad \sigma^2 = np(1-p) = \frac{5000}{36} \qquad \sigma = \frac{50\sqrt{2}}{6}$$

We can approximate this with a normal random variable X with μ and σ^2 as parameters. We want between 150 and 200 throws of 6 (inclusively). We use continuity correction:

$$\begin{aligned}
\mathbf{P}\{150 \le X \le 200\} &= \mathbf{P}\{149.5 < X < 200.5\} \\
&= \mathbf{P}\{X < 200.5\} - \mathbf{P}\{X < 149.5\} \\
&= \Phi\Big(\frac{200.5 - (1000/6)}{(50\sqrt{2}/6)}\Big) - \Phi\Big(\frac{149.5.5 - (1000/6)}{(50\sqrt{2}/6)}\Big) \\
&\approx \Phi(2.87) - \Phi(-1.46) = \Phi(2.87) + \Phi(1.46) - 1 \\
&\approx 0.9979 + 0.9279 - 1 \\
&\approx 0.9258.
\end{aligned}$$

23b. Suppose 6 appears 200 times out of 1000 throws. This leaves 800 remaining throws on which we could get a 5. On these throws, the only values that can appear are 1,2,3,4,5, and since the throws are independent, the probability of throwing a 5 on any throw is $\frac{1}{5} = 0.2$. We want the probability of throwing fewer than 150 throws. This is a binomial random variable with $n = 800$ and $p = 0.2$ (six will not appear). So,

$$\mu = np = 800(0.2) = 160 \qquad \sigma^2 = np(1-p) = 800(0.2)(0.8) = 128 \qquad \sigma = 8\sqrt{2}$$

We can approximate this with a normal random variable X with μ and σ^2 as parameters. We want fewer than 150 throws of 5. We use continuity correction:

$$\begin{aligned}\mathbf{P}\{X < 150\} &= \mathbf{P}\{X < 149.5\} \\ &= \Phi\left(\frac{149.5 - 160}{8\sqrt{2}}\right) \\ &= 1 - \Phi\left(\frac{10.5}{8\sqrt{2}}\right) \\ &\approx 1 - \Phi(0.93) \\ &\approx 1 - 0.8238 \\ &\approx 0.1762.\end{aligned}$$

25. Let X denote the number of unacceptable items. Then X is well approximated by a binomial random variable with $n = 150$ and $p = 0.05$. So,

$$\mu = np = 150(0.05) = 7.5 \quad \sigma^2 = np(1-p) = 150(0.05)(0.95) = 7.125 \quad \sigma = \sqrt{7.125}$$

We approximate X with a normal distribution using these parameters. We want no more than 10 defective items. Using continuity correction

$$\begin{aligned}\mathbf{P}\{X \leq 10\} &= \mathbf{P}\{X < 10.5\} \\ &= \Phi\left(\frac{10.5 - 7.5}{\sqrt{7.125}}\right) \\ &= \Phi(1.12) \\ &\approx 0.8686\end{aligned}$$

26a. With a fair coin, let X count the number of heads. X is a binomial random variable with parameters $n = 1000$ and $p = 0.5$. So,

$$\mu = np = 1000(0.5) = 500 \quad \sigma^2 = np(1-p) = 1000(0.5)^2 = 250 \quad \sigma = 5\sqrt{10}$$

We approximate X with a normal distribution using these parameters. We reach a false conclusion if there are at least 525 heads when using a fair coin.

Using continuity correction

$$\begin{aligned}
\mathbf{P}\{525 \le X\} &= \mathbf{P}\{524.5 < X\} \\
&= 1 - \mathbf{P}\{X \le 524.5\} \\
&= 1 - \Phi\left(\frac{524.5 - 500}{5\sqrt{10}}\right) \\
&\approx 1 - \Phi(1.55) \\
&\approx 1 - 0.9394 \\
&\approx 0.0606.
\end{aligned}$$

26b. With a biased coin, let Y count the number of heads. X is a binomial random variable with parameters $n = 1000$ and $p = 0.55$. So,

$$\mu = np = 1000(0.55) = 550 \quad \sigma^2 = np(1-p) = 1000(0.55)(0.45) = 247.5 \quad \sigma = \sqrt{247.5}$$

We approximate Y with a normal distribution using these parameters. We reach a false conclusion if there are fewer than 525 heads when using a biased coin. Using continuity correction

$$\begin{aligned}
\mathbf{P}\{Y < 525\} &= \mathbf{P}\{Y \le 524.5\} \\
&= \mathbf{P}\{Y < 524.5\} \\
&= \Phi\left(\frac{524.5 - 550}{\sqrt{247.5}}\right) \\
&\approx \Phi(-1.62) \\
&\approx 1 - \Phi(1.62) \\
&\approx 1 - 0.9474 \\
&\approx 0.0526.
\end{aligned}$$

28. The probability that there are at least 20 lefthanded students in a school of 200 students is directly given by a hypergeometric random variable. But since the general population is so large compared to that of the school, we can approximate this random variable by a binomial random variable with parameters $n = 200$ and $p = 0.12$ (provided that the students are randomly drawn from the general population – that is, this is not a school for lefthanders, for example).

Since a binomial random variable can be approximated by a normal random variable, we use this with parameters

$$\mu = np = 24 \quad \sigma^2 = \sigma = np(1-p) \approx 21.12 \quad \sqrt{np(1-p)} \approx 4.6.$$

Let X be a normally distributed random variable with these parameters. We want at least 20 lefthanded students. We apply continuity correction

$$\begin{aligned}
\mathbf{P}\{20 \le X\} &= \mathbf{P}\{19.5 < X\} \\
&= \mathbf{P}\{19.5 < (4.6)Z + 24\} \\
&= \mathbf{P}\{\frac{-4.5}{4.6} \le Z\} \\
&= \Phi(\frac{4.5}{4.6}) \\
&\approx \Phi(0.98) \\
&\approx 0.8365
\end{aligned}$$

32a. Let X be the time in hours required to repair the machine. So, X is exponentially distributed with parameter $\frac{1}{2}$ (the average time to repair the machine is 2 hours).

The probability that a repair time takes longer than two hours is

$$\begin{aligned}
\mathbf{P}\{2 < X\} &= 1 - \mathbf{P}\{X \le 2\} \\
&= 1 - F_X(2) \\
&= 1 - \left(1 - e^{-1}\right) = e^{-1} \approx 0.368
\end{aligned}$$

32b. The probability that a repair requires more than 10 hours, given that its duration exceeds 9 hours is

$$\begin{aligned}
\mathbf{P}(10 < X \mid 9 < X) &= \mathbf{P}\{1 < X\} = 1 - \mathbf{P}\{X \le 1\} \\
&= 1 - F_X(1) \\
&= 1 - (1 - e^{-0.5}) = e^{-0.5} \\
&\approx 0.6065.
\end{aligned}$$

This method used the fact that the exponential distribution is memoryless.

An alternative method is to simply compute the conditional probability directly:

$$\begin{aligned}
\mathbf{P}(10 < X \,|\, 9 < X) &= \frac{\mathbf{P}\{10 < X, 9 < X\}}{\mathbf{P}\{9 < X\}} \\
&= \frac{1 - \mathbf{P}\{X \le 10\}}{1 - \mathbf{P}\{X \le 9\}} \\
&= \frac{1 - F_X(10)}{1 - F_X(9)} \\
&= \frac{1 - (1 - e^{-5})}{1 - (1 - e^{-4.5})} \\
&= e^{-0.5}.
\end{aligned}$$

33. Let X be the random variable giving the years a radio functions. X is exponentially distributed with parameter $\lambda = \frac{1}{8}$ (the expected lifetime of the radio is 8 years). We do not know how old the radio is, but since X is memoryless, the probability that the radio lasts an additional 8 years is

$$\begin{aligned}
\mathbf{P}\{8 < X\} &= 1 - \mathbf{P}\{X \le 8\} \\
&= 1 - F_X(8) \\
&= 1 - (1 - e^{-(1/8)8}) = e^{-1} \\
&\approx 0.368.
\end{aligned}$$

34a. Let X be the random variable for the total number of miles in the lifetime of a car (in thousands of miles). X is an exponentially distributed random variable with parameter $\lambda = \frac{1}{20}$ (the expected number of miles is 20 thousand).

The probability that a car with 10 thousand miles is driven another 20 thousand miles is

$$\begin{aligned}
\mathbf{P}(30 < X \,|\, 10 < X) &= \mathbf{P}\{20 < X\} = 1 - \mathbf{P}\{X \le 20\} \\
&= 1 - F_X(20) \\
&= 1 - (1 - e^{-(1/20)20} = e^{-1} \\
&\approx 0.368.
\end{aligned}$$

This method used the fact that the exponential distribution is memoryless.

An alternative method is to simply compute the conditional probability directly:

$$\begin{aligned}
\mathbf{P}(30 < X \mid 10 < X) &= \frac{\mathbf{P}\{30 < X, 10 < X\}}{\mathbf{P}\{10 < X\}} \\
&= \frac{1 - \mathbf{P}\{X \le 30\}}{1 - \mathbf{P}\{X \le 10\}} \\
&= \frac{1 - F_X(30)}{1 - F_X(10)} \\
&= \frac{1 - (1 - e^{-(1/20)30})}{1 - (1 - e^{-(1/20)10})} \\
&= e^{-1}.
\end{aligned}$$

34b. Now suppose the random variable X is uniformly distributed on the interval $[0, 40]$. The probability that the car is driven another 20 thousand miles, given it has been 10 thousand miles already is

$$\begin{aligned}
\mathbf{P}(30 < X \mid 10 < X) &= \frac{\mathbf{P}\{30 < X, 10 < X\}}{\mathbf{P}\{10 < X\}} \\
&= \frac{1 - \mathbf{P}\{X \le 30\}}{1 - \mathbf{P}\{X \le 10\}} \\
&= \frac{1 - F_X(30)}{1 - F_X(10)} \\
&= \frac{0.25}{0.75} \\
&= \frac{1}{3} \approx 0.333
\end{aligned}$$

37a. Let X be uniformly distributed on $(-1, 1)$.

$$\begin{aligned}\mathbf{P}\{|X| > \frac{1}{2}\} &= \mathbf{P}\{X < -\frac{1}{2}\} + \mathbf{P}\{X > \frac{1}{2}\} \\ &= F_X(-\frac{1}{2}) + (1 - F_X(\frac{1}{2}) \\ &= \int_{-1}^{-1/2} \frac{1}{2}\, dt + \big(1 - \int_{-1}^{1/2} \frac{1}{2}\, dt\big) \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.\end{aligned}$$

37b. Let $Y = |X|$. First compute the cumulative distribution $F_Y(a)$.

$$\begin{aligned}F_Y(a) &= \mathbf{P}\{Y \leq a\} \\ &= \mathbf{P}\{|X| \leq a\} \\ &= \mathbf{P}\{-a \leq X \leq a\} \\ &= F_X(a) - F_X(-a)\end{aligned}$$

Now compute the density $f_Y(a)$.

$$\begin{aligned}f_Y(a) &= \frac{d}{da} F_Y(a) \\ &= \frac{d}{da}\big(F_X(a) - F_X(-a)\big) \\ &= f_X(a) + f_X(-a).\end{aligned}$$

Since X is uniformly distributed on $(-1, 1)$,

$$f_X(a) = \begin{cases} \frac{a+1}{2} & \text{if } a \in (-1,1) \\ 0 & \text{otherwise .} \end{cases}$$

So, $f_Y(a) = \frac{a+1}{2} + \frac{-a+1}{2} = 1$ when $a \in (-1, 1)$, thus

$$f_Y(a) = \begin{cases} 1 & \text{if } a \in (-1,1) \\ 0 & \text{otherwise .} \end{cases}$$

Intuitively, if X is uniformly distributed on $(-1, 1)$ then $|X|$ is uniformly distributed on $[0, 1)$.

38. The key is to determine the range of values to $a \in (0,5)$ such that there are two real solutions to the equation

$$4x^2 + 4ax + (a+2) = 0.$$

The solutions to this equation are given by the quadratic equation

$$\frac{-8a \pm \sqrt{16a^2 - 16(a+2)}}{8}.$$

There are two real roots when the discriminant is greater than zero:

$$16a^2 - 16(a+2) > 0 \qquad \text{equvalently, } a^2 - a - 2 > 0.$$

The roots of $a^2 - a - 2 = (a-2)(a+1)$ are $a = -1$ and $a = 2$, so we need to evaluate the behavior of this polynomial on the intervals

$$\begin{array}{ll} 0 < a < 2 & a^2 - a - 2 < 0 \\ 2 < a < 5 & a^2 - a - 2 > 0 \end{array}$$

So, the polynomial $4x^2 + 4ax + (a+2)$ has real roots when $2 < a < 5$ for $a \in (0,5)$.

Since Y is uniformly distributed on $(0,5)$,

$$\mathbf{P}\{2 < Y < 5\} = \int_2^5 \frac{1}{5}\, dt = \frac{3}{5}.$$

39. X is an exponential random variable with parameter $\lambda = 1$. Let $Y = \log X$. The cumulative distribution for Y is

$$\begin{aligned} F_Y(t) &= \mathbf{P}\{Y \le t\} \\ &= \mathbf{P}\{\log X \le t\} \\ &= \mathbf{P}\{X \le e^t\} \\ &= 1 - e^{-e^t} \end{aligned}$$

The density of Y is obtained by differentiating

$$\begin{aligned} f_Y(t) &= \frac{d}{dt}\left(1 - e^{-e^t}\right) \\ &= e^t e^{-e^t} \\ &= e^{t - e^t} \end{aligned}$$