MATH 425
HOMEWORK 6
Winter, 2009

1a. Since $\mathbf{P}\{-1 < X < 1\} = 1$, we compute

$$\begin{aligned}\int_{-1}^{1} c(1-x^2)\,dx &= c(x - \frac{x^3}{3})\Big|_{x=-1}^{-1} \\ &= c\frac{2}{3} + c\frac{2}{3} = \frac{4}{3}c.\end{aligned}$$

So, $c = \frac{3}{4}$.

1b. The cumulative distribution depends on three cases:

$$F(a) = \mathbf{P}\{X \leq a\} = \begin{cases} 0 & \text{if } a < 1 \\ \frac{3a-a^3+2}{4} & \text{if } -1 \leq a \leq 1 \\ 1 & \text{if } a > 1. \end{cases}$$

Here is the computation of the case when $-1 \leq a \leq 1$:

$$\begin{aligned}\frac{3}{4}\int_{-1}^{a} (1-x^2)\,dx &= \frac{3}{4}\big(x - \frac{x^3}{3}\big)\Big|_{x=-1}^{a} \\ &= \frac{3}{4}\big(a - \frac{a^3}{3} + \frac{2}{3}\big) \\ &= \frac{3a - a^3 + 2}{4}.\end{aligned}$$

2. The first task is to compute C. Since $\mathbf{P}\{0 < X\} = 1$, we compute the integral using separation by parts:

$$\begin{aligned}\int_{0}^{\infty} Cxe^{-\frac{x}{2}}\,dx &= -2Cxe^{-\frac{x}{2}}\Big|_{x=0}^{\infty} + \int_{0}^{\infty} 2Ce^{-\frac{x}{2}}\,dx \\ &= -4Ce^{-\frac{x}{2}}\Big|_{x=0}^{\infty} \\ &= 4C\end{aligned}$$

So, $C = \frac{1}{4}$.

The probability that the system works at least 5 months is $\mathbf{P}\{X \geq 5\}$:

$$\begin{aligned}
\frac{1}{4}\int_5^{\infty} xe^{-\frac{x}{2}}\,dx &= -\frac{1}{2}xe^{-\frac{x}{2}}\Big|_{x=5}^{\infty} + \frac{1}{2}\int_5^{\infty} e^{-\frac{x}{2}}\,dx \\
&= \frac{5}{2}e^{-\frac{5}{2}} - \left(e^{-\frac{x}{2}}\right)\Big|_{x=5}^{\infty} \\
&= \frac{5}{2}e^{-\frac{5}{2}} + e^{-\frac{5}{2}} = \frac{7}{2}e^{-\frac{5}{2}}.
\end{aligned}$$

So, $\mathbf{P}\{X \geq 5\} = \frac{7}{2}e^{-\frac{5}{2}}$.

4a. The probability the device works for more than 20 hours, $\mathbf{P}\{X > 20\}$, is

$$\int_2 0^{\infty} 10x^{-2}\,dx = -10x^{-1}\Big|_{x=20}^{\infty} = \frac{1}{2}.$$

So, $\mathbf{P}\{X > 20\} = \frac{1}{2}$.

4b. The cumulative distribution function depends on two cases

$$F(a) = \mathbf{P}\{X \leq a\} = \begin{cases} 0 & \text{if } a \leq 10 \\ 1 - \frac{10}{a} & \text{if } 10 < a \end{cases}$$

Here is the computation of the case when $10 < a$:

$$\int_{10}^{a} 10x^{-2}\,dx = -10x^{-1}\Big|_{x=10}^{a} = 1 - \frac{10}{a}$$

4c. We will assume that the time one of the devices works is independent of any other device. The probability that one device works from part (4b) is

$$\mathbf{P}\{X \geq 15\} = \mathbf{P}\{X > 15\} = 1 - F(15) = \frac{2}{3}.$$

The probability that at least 3 out of 6 will function is a binomial random variable Y:

$$\begin{aligned}
\mathbf{P}\{Y \geq 3\} &= \mathbf{P}\{Y = 3\} + \mathbf{P}\{Y = 4\} + \mathbf{P}\{Y = 5\} + \mathbf{P}\{Y = 6\} \\
&= \binom{6}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 + \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + \binom{6}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1 + \binom{6}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0 \\
&\approx 0.8999.
\end{aligned}$$

7. We have two unknowns, a and b, and two constraints. First, $\mathbf{P}\{0 \leq X \leq 1\} = 1$:

$$\begin{aligned}
\mathbf{P}\{0 \leq X \leq 1\} &= \int_0^1 a + bx^2\,dx \\
&= \left(ax + \frac{bx^3}{3}\right)\Big|_{x=0}^{1} \\
&= a + \frac{b}{3};
\end{aligned}$$

and second, $E[X] = \frac{3}{5}$:

$$\begin{aligned}
E[X] &= \int_0^1 ax + bx^3\,dx \\
&= \left(\frac{ax^2}{2} + \frac{bx^4}{4}\right)\Big|_{x=0}^{1} \\
&= \frac{a}{2} + \frac{b}{4}.
\end{aligned}$$

So, we have two equations and two unknowns:

$$\begin{aligned}
1 &= a + \frac{b}{3} \\
\frac{3}{5} &= \frac{a}{2} + \frac{b}{4}
\end{aligned}$$

Solving these equations gives

$$a = \frac{3}{5} \qquad b = \frac{6}{5}.$$

8. The expected lifetime of the tube is 2 hours:

$$\begin{aligned}
\int_0^\infty x^2 e^{-x}\,dx &= -x^2e^{-x}\Big|_{x=0}^{\infty} + 2\int_0^\infty xe^{-x}\,dx \\
&= -2xe^{-x}\Big|_{x=0}^{\infty} + 2\int_0^\infty e^{-x}\,dx \\
&= -2e^{-x}\Big|_{x=0}^{\infty} = 2.
\end{aligned}$$

(Use integration by parts twice.)

10a. The probability density function for the random variable X giving the arrival time (in hours) is

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The passenger takes the A-train if he arrives in the intervals 7:05-7:15, 7:20-7:30, 7:35-7:45, or 7:50-8:00. Each time interval is equally likely, so we can lump the time into 40 minutes out of 60 that he can take take the A-train. So, the probability of catching the A-train is $\frac{2}{3}$.

10b. The passenger now takes the A-train if he arrives in the intervals: 7:10-7:15, 7:20-7:30, 7:35-7:45, 7:50-8:00, 8:05-8:10. This again sums to 40 minutes out of the 60 minute time period of possible arrival times. So, the probability of catching the A-train is $\frac{2}{3}$.

11. We interpret the statement to mean that the random variable X giving the point chosen has a uniform probability density function:

$$f(x) = \begin{cases} c & \text{if } 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathbf{P}\{0 \le X \le L\} = 1$, we must have $c = \frac{1}{L}$.

Let s be the length of the shorter segment and $\ell = L - s$ the length of the longer segment. Then

$$\frac{s}{\ell} < \frac{1}{4} \quad \text{when} \quad 0 \le s < \frac{L}{5} \text{ or } \frac{4L}{5} < s \le L.$$

So, the probability that $\frac{s}{\ell} < \frac{1}{4}$ is

$$\frac{1}{L}\int_0^{\frac{L}{5}} dx + \frac{1}{L}\int_{\frac{4L}{5}}^{L} dx \quad = \quad \frac{x}{L}\Big|_{x=0}^{\frac{L}{5}} + \frac{x}{L}\Big|_{x=\frac{4L}{5}}^{L} = \frac{2}{5}.$$

13a. Let X be the random variable giving the time of the bus arrival in minutes. The probability density function is given by

$$f(x) = \begin{cases} \frac{1}{30} & \text{if } 0 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}$$

You must wait longer than ten minutes if the bus arrives after 10:10. The probability is

$$\frac{1}{30}\int_{10}^{30} dx = \frac{20}{30}.$$

The probability of waiting longer than ten minutes is $\frac{2}{3}$.

13b. In order to wait an additional ten minutes, the bus must arrive between 10:25 and 10:30. We want the conditional probability

$$\mathbf{P}(10{:}25 < X \,|\, 10{:}15 < X) \quad = \quad \frac{\frac{1}{30}\int_{25}^{30} dx}{\frac{1}{30}\int_{15}^{30} dx} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

16. Let X be the random variable counting rainfall (in inches) in one year. Since X is normally distributed with $\mu = 40$ and $\sigma = 4$, we can compute $\mathbf{P}\{X > 50\}$ by standardizing :

$$\mathbf{P}\{X > 50\} = \mathbf{P}\{\frac{X-40}{4} > \frac{50-40}{4}\} = 1 - \mathbf{P}\{\frac{X-40}{4} \le 2.5\} = 1 - \Phi(2.5) \approx 0.0062.$$

We will assume that the amount of rain falling in a given year is independent of the amount fallling in any other year. So, the probability that no year in the next ten has more that 50 inches of rainfall is

$$\big(1 - \mathbf{P}\{X > 50\}\big)^{10} \approx \big(0.9938\big)^{10} \approx 0.9397$$

19. X is a normal random variable with $\mu = 12$ and $\sigma = 2$. We compute

$$0.10 = \mathbf{P}\{X > c\} = \mathbf{P}\{\frac{X-12}{2} > \frac{c-12}{2}\} = 1 - \Phi(\frac{c-12}{2})$$

We want c so that $\Phi(\frac{c-12}{2}) = 0.90$. Thus, $\frac{c-12}{2} \approx 1.28$, and $c \approx 14.56$.

22a. Let X be the normally distributed r.v. with $\mu = 0.9000$ and $\sigma = 0.0030$ giving the width of a slot of a duralumin forging (in inches). Since the specification is 0.9000 ± 0.0050, a forging will be defective if $X < 0.8950$ or

$X > 0.9050$. This probability is

$$\begin{aligned}
\mathbf{P}\{X < 0.8950\} + \mathbf{P}\{X > 0.9050\} &= 2 \cdot \mathbf{P}\{X > 0.9050\} \\
&= 2 \cdot \mathbf{P}\{\frac{X - 0.9000}{0.0030} > \frac{0.9050 - 0.9000}{0.0030}\} \\
&= 2 \cdot \big(1 - \Phi(1.67)\big) \\
&\approx 0.0950
\end{aligned}$$

The first equality is because a normally distributed variable is symmetric about its mean (see Ross, p. 221).

22b. We now want to find σ so that (see problem 22a)

$$0.01 \geq 2 \cdot \mathbf{P}\{\frac{X - 0.9000}{\sigma} > \frac{0.9050 - 0.9000}{\sigma}\} = 2 \cdot \big(1 - \Phi(\frac{0.0050}{\sigma})\big)$$

Equivalently, $\Phi(\frac{0.0050}{\sigma}) \geq 0.9950$. Since the cumulative distribution Φ is increasing, the inequality holds when

$$\frac{0.0050}{\sigma} \geq 2.58 \quad \text{or} \quad \sigma \leq 0.0019.$$