MATH 425
HOMEWORK 4
Winter, 2009

1 Chapter 3

56. Let N be the event that the nth coupon collected is new, and let E_i be the event that the nth coupon collected is type i. By conditioning on the events E_i,

$$\mathbf{P}(N) = \sum_{i=1}^{m} \mathbf{P}(N \mid E_i) \cdot \mathbf{P}(E_i).$$

We are given $\mathbf{P}(E_i) = p_i$. Suppose coupon i is first collected on the nth coupon. Then, the previous $n-1$ coupons are something other than the ith. The probabiltiy that a coupon collected is not the ith is $1 - p_i$, and since each coupon collected is independent of any other coupon collected,

$$\mathbf{P}(N \mid E_i) = (1 - p_i)^{n-1}.$$

So,

$$\mathbf{P}(N) = \sum_{i=1}^{m} p_i (1 - p_i)^{n-1}.$$

74. The probability that A throws a nine before B throws a six, when A throws first, is $\frac{9}{19}$.

This problem is nearly identical to a problem I did in Lecture 12. D_6 (D_9) be the event that the die comes-up six (nine) on the first roll, and D that the die comes-up some other value. So,

$$\mathbf{P}(D_6) = \frac{5}{36} \quad \mathbf{P}(D_9) = \frac{4}{36} = \frac{1}{9}$$

There are two methods.

Method 1. Let A_{2n+1} be the event that A wins on the $2n+1$st throw (where $n \geq 0$). So,

$$\mathbf{P}(A_{2n+1} = \left(\frac{32}{36}\right)^n \cdot \left(\frac{31}{36}\right)^n \cdot \frac{1}{9},$$

since A must throw a nine on the $2n+1$st throw, but must throw something besides a nine on the n other odd throws, and B must throw something besides a six on the n even throws.

The events A_{2n+1} are mutually exclusive, so the probability that A wins on some throw (which must be odd) is

$$\begin{aligned}\mathbf{P}(\bigcup_{n=0}^{\infty} A_{2n+1}) &= \sum_{n=0}^{\infty} (\frac{32}{36})^n \cdot (\frac{31}{36})^n \cdot \frac{1}{9} \\ &= \frac{1}{9} \cdot \sum_{n=0}^{\infty} (\frac{32}{36}\frac{31}{36})^n \\ &= \frac{1}{9} \cdot \frac{1}{1-\frac{32\cdot 31}{36^2}} \\ &= \frac{9}{19}\end{aligned}$$

Method 2. We use conditioning and independence. There are three possible outcomes in the next two throws:

1. A throws a nine (so, A wins),
2. A throws something besides a nine and B throws a six (so, A cannot win),
3. A throws something besides a nine and B throw something besides a six and A eventually wins after the second throw.

Note that by independence, the probability that A eventually wins is the same as the probability that A eventually wins given neither A nor B gets a winning throw on their first turns.

Let A be the event that A eventually wins. Then, using cases 1 and 2 above, together with independence:

$$\mathbf{P}(A) = \frac{1}{9} + \frac{8}{9} \cdot \frac{31}{36} \cdot \mathbf{P}(A).$$

Solving for $\mathbf{P}(A)$ we have

$$\mathbf{P}(A) = \frac{1}{9} \cdot \frac{1}{1-\frac{8}{9}\cdot\frac{31}{36}} = \frac{9}{19}.$$

77.

Let E_1 (E_2) be the events that that outcome 1 occurs in trial 1 (trial 2). Let F be the event that outcome 3 occurs last.

(a). We want to compute

$$\mathbf{P}(E_1 \mid F) = \mathbf{P}(F \mid E_1) \cdot \frac{\mathbf{P}(E_1)}{\mathbf{P}(F)}$$

We have the following probabilities:

$$\mathbf{P}(E_1) = \frac{1}{3} \quad \mathbf{P}(F \mid E_1) = \frac{1}{2} \quad \mathbf{P}(F) = \frac{1}{3}$$

The first is given by the problem. Since outcome 3 is as equally likely to be last as outcomes 1 or 2 (or outcome 2, given outcome 1 occurs on the first trial), the other two probabilities are clear. (I give a more explicit argument below.)

So,

$$\mathbf{P}(E_1 \mid F) = \frac{1}{2} \cdot \frac{\frac{1}{3}}{\frac{1}{3}} = \frac{1}{2}.$$

(b). We want to compute

$$\mathbf{P}(E_1 \cap E_2 \mid F) = \mathbf{P}(F \mid E_1 \cap E_2) \cdot \frac{\mathbf{P}(E_1 \cap E_2)}{\mathbf{P}(F)}$$

We have the following additional probabilities:

$$\mathbf{P}(E_1 \cap E_2) = \mathbf{P}(E_1) \cdot \mathbf{P}(E_2) = \frac{1}{9} \quad \mathbf{P}(F \mid E_1 \cap E_2) = \frac{1}{2}$$

The first is given by the problem, and the second is since outcome 3 is as equally likely to occur as outcome 2 after outcome 1 occurs on the first two trials.

So,

$$\mathbf{P}(E_1 \cap E_2 \mid F) = \frac{1}{2} \cdot \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{6}.$$

Here is a more explicit argument for $\mathbf{P}(F) = \frac{1}{3}$. The idea is along the lines of Lecture 12. Let F_1 (G_1) be outcome 3 (outcome 2) occurs in the first trial. By conditioning and independence:

$$\begin{aligned}
\mathbf{P}(F) &= \mathbf{P}(F \mid E_1) \cdot \mathbf{P}(E_1) + \mathbf{P}(F \mid F_1) \cdot \mathbf{P}(F_1) + \mathbf{P}(F \mid G_1) \cdot \mathbf{P}(G_1) \\
&= \frac{1}{3} \cdot \mathbf{P}(F \mid E_1) + 0 + \frac{1}{3} \cdot \mathbf{P}(F \mid G_1) \\
&= \frac{2}{3} \cdot \mathbf{P}(F \mid E_1) \qquad \text{since } \mathbf{P}(F \mid E_1) = \mathbf{P}(F \mid G_1)
\end{aligned}$$

The last line is by independence and the symmetry of the situations (i) when outcome 1 occurs in the first trial and (ii) when outcome 2 occurs in the first trial. (That is, you can reverse the rolls of E_1 and G_1 without changing the probability. As I show below, $\mathbf{P}(F \mid E_1) = \frac{1}{2} = \mathbf{P}(F \mid G_1)$.)

Let E_2 (F_2, G_2) be the event that outcome 1 (outcome 3, outcome 2) occurred in the second trial. Let $\mathbf{P}_{E_1}(()\cdot) = \mathbf{P}(\cdot \mid E_1)$. By conditioning and independence:

$$\begin{aligned}
\mathbf{P}_{E_1}(F) &= \mathbf{P}_{E_1}(F \mid E_2) \cdot \mathbf{P}_{E_1}(E_2) + \mathbf{P}_{E_1}(F \mid F_2) \cdot \mathbf{P}_{E_1}(F_2) + \mathbf{P}_{E_1}(F \mid G_2) \cdot \mathbf{P}_{E_1}(G_2) \\
&= \frac{1}{3} \cdot \mathbf{P}_{E_1}(F \mid E_2) + 0 + \frac{1}{3} \cdot 1
\end{aligned}$$

By independence, $\mathbf{P}_{E_1}(F) = \mathbf{P}_{E_1}(F \mid E_2)$, since we still need outcome 2 to occur before outcome 3, and this probability is independent of outcome 1 occurring in trials 1 and 2. So,

$$\mathbf{P}_{E_1}(F) = \frac{1}{3} \cdot \mathbf{P}_{E_1}(F) + 0 + \frac{1}{3},$$

and solving for $\mathbf{P}_{E_1}(F)$

$$\mathbf{P}_{E_1}(F) = \frac{1}{2}.$$

Since $\mathbf{P}_{E_1}(F) = \mathbf{P}(F \mid E_1)$, we can solve back for $\mathbf{P}(F)$:

$$\mathbf{P}(F) = \frac{2}{3} \cdot \mathbf{P}(F \mid E_1) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

2 Chapter 4

4. The easiest way to compute the probabilities is to use conditioning (the Multiplication Rule). For example the probability that males occupy the first two positions, and a female the third, can be computed by

$$\mathbf{P}(M_1 \cap M_2 \cap F_3) = \mathbf{P}(M_1) \cdot \mathbf{P}(M_2 \,|\, M_1) \cdot \mathbf{P}(F_3 \,|\, M_1 \cap M_2) = \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{5}{8}$$

So,

$$\begin{aligned}
\mathbf{P}\{X = 1\} &= \frac{5}{10} = \frac{1}{2} \\
\mathbf{P}\{X = 2\} &= \frac{5}{10} \cdot \frac{5}{9} = \frac{5}{18} \\
\mathbf{P}\{X = 3\} &= \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{5}{8} = \frac{5}{36} \\
\mathbf{P}\{X = 4\} &= \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{4}{7} = \frac{5}{84} \\
\mathbf{P}\{X = 5\} &= \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} \cdot 56 = \frac{5}{252} \\
\mathbf{P}\{X = 6\} &= \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} \cdot 36 \cdot 55 = \frac{1}{252} \\
\mathbf{P}\{X \geq 7\} &= 0
\end{aligned}$$

9.

$$\begin{aligned}
\mathbf{P}\{X = -2\} &= \frac{8 \cdot 8}{14^2} = \frac{16}{49} \\
\mathbf{P}\{X = -1\} &= \frac{2^2 \cdot 8}{14^2} = \frac{8}{49} \\
\mathbf{P}\{X = 0\} &= \frac{2^2}{14^2} = \frac{1}{49} \\
\mathbf{P}\{X = 1\} &= \frac{2 \cdot 8 \cdot 4}{14^2} = \frac{16}{49} \\
\mathbf{P}\{X = 2\} &= \frac{2^2 \cdot 4}{14^2} = \frac{4}{49} \\
\mathbf{P}\{X = 4\} &= \frac{4 \cdot 4}{14^2} = \frac{4}{49}
\end{aligned}$$

14. Let $(v_1, v_2, v_3, v_4, v_5)$ be the assignment of values to the five players. There are $5! = 120$ assignments.

To compute $\mathbf{P}\{X = 1\}$, for example, we must compute $\mathbf{P}(v_3 > v_1 > v_2)$:

- $v_1 = 2$: $v_2 = 1$, $v_3 = 3, 4, 5$. So, $3 \cdot 2!$ possibilities (v_4, v_5 can be anything).
- $v_1 = 3$: $v_2 = 1, 2$, $v_3 = 4, 5$. So, $4 \cdot 2!$ possibilities.
- $v_1 = 4$: $v_2 = 1, 2, 3$, $v_3 = 5$. So, $3 \cdot 2!$ possibilities

The rest of the cases are computed similarly.

$$\begin{aligned}
\mathbf{P}\{X = 0\} &= \frac{(4+3+2+1)\cdot 3!}{5!} = \frac{1}{2} \\
\mathbf{P}\{X = 1\} &= \frac{(3+4+3)\cdot 2!}{5!} = \frac{1}{6} \\
\mathbf{P}\{X = 2\} &= \frac{(4+6)}{5!} = \frac{1}{12} \\
\mathbf{P}\{X = 3\} &= \frac{6}{5!} = \frac{1}{20} \\
\mathbf{P}\{X = 4\} &= \frac{4!}{5!} = \frac{1}{5}.
\end{aligned}$$

19. The probability mass function for X is

$$\begin{aligned}
p(0) &= \frac{1}{2} \\
p(1) &= \frac{1}{10} \\
p(2) &= \frac{1}{5} \\
p(3) &= \frac{1}{10} \\
p(3.5) &= \frac{1}{10}
\end{aligned}$$

20a. Here is the probability mass function associated with the random variable X.

$$\begin{aligned}
\mathbf{P}\{X=-3\} &= \left(\frac{20}{38}\right)^2 \\
\mathbf{P}\{X=-1\} &= 2\cdot\frac{18}{38}\cdot\left(\frac{20}{38}\right)^2 \\
\mathbf{P}\{X=1\} &= \frac{18}{38}+\left(\frac{10}{38}\right)^2\cdot\frac{20}{38}
\end{aligned}$$

So, $\mathbf{P}\{X>0\}=\mathbf{P}\{X=1\}\approx 0.5918$.

20c. See **20a** for the probabilities.

$$E[X]=(-3)\cdot\mathbf{P}\{X=-3\}+(-1)\cdot\mathbf{P}\{X=-1\}+(1)\cdot\mathbf{P}\{X=1\}=-0.108$$

22b. Let X be the number of games played in the best-of-five format. So, $X=3,4,5$. Team A wins with probability p and Team B wins with probability $q=(1-p)$. In computing the probabilities, remember the final game is won by the series winner, so the only remaining choices is the order winner in the first games.

$$\begin{aligned}
\mathbf{P}\{X=3\} &= p^3+q^3 \\
\mathbf{P}\{X=4\} &= \binom{3}{1}p^3q+\binom{3}{1}pq^3 \\
\mathbf{P}\{X=5\} &= \binom{4}{2}p^3q^2+\binom{4}{2}p^2q^3
\end{aligned}$$

So, the expected number of wins is

$$\begin{aligned}
E[X] &= 3\mathbf{P}\{X=3\}+4\mathbf{P}\{X=4\}+5\mathbf{P}\{X=5\} \\
&= 3(p^3+q^3)+4(3p^3q+3pq^3)+5(6p^3q^2+6p^2q^3) \\
&= 3p^3(1+4q+10q^2)+3q^3(1+4p+10p^2)
\end{aligned}$$

When $p=\frac{1}{2}$,

$$E[X]=\frac{33}{8}=4.125$$

25. Let X be the random variable of the winnings on a single play of the slot machine.

$$\begin{aligned}
\mathbf{P}\{X = 60\} &= \frac{3}{20^3} \\
\mathbf{P}\{X = 20\} &= \frac{12}{20^3} \\
\mathbf{P}\{X = 18\} &= \frac{4}{20^3} \\
\mathbf{P}\{X = 14\} &= \frac{24}{20^3} \\
\mathbf{P}\{X = 12\} &= \frac{126}{20^3} \\
\mathbf{P}\{X = 8\} &= \frac{21}{20^3} \\
\mathbf{P}\{X = 2\} &= \frac{980}{20^3} \\
\mathbf{P}\{X = 0\} &= \frac{1820}{20^3} \\
\mathbf{P}\{X = -1\} &= \frac{5010}{20^3}
\end{aligned}$$

So, the expected winnings on a single play is

$$\begin{aligned}
E[X] &= (-1)\frac{5010}{20^3} + (2)\frac{980}{20^3} + (8)\frac{21}{20^3} + (12)\frac{126}{20^3} + (14)\frac{24}{20^3} \\
&\quad (18)\frac{4}{20^3} + (20)\frac{12}{20^3} + (60)\frac{3}{20^3} \\
&= -\frac{271}{4000} \approx -0.6775.
\end{aligned}$$

28. Let X be the number of defects in the three sampled.

$$\begin{aligned}
\mathbf{P}\{X = 0\} &= \frac{16 \cdot 15 \cdot 14}{20 \cdot 19 \cdot 18} = \frac{28}{57} \\
\mathbf{P}\{X = 1\} &= \binom{3}{1} \cdot \frac{16 \cdot 15 \cdot \cdot 4}{20 \cdot 19 \cdot 18} = \frac{8}{19} \\
\mathbf{P}\{X = 2\} &= \binom{3}{2} \cdot \frac{16 \cdot 4 \cdot \cdot 3}{20 \cdot 19 \cdot 18} = \frac{8}{95} \\
\mathbf{P}\{X = 3\} &= \frac{4 \cdot 3 \cdot \cdot 2}{20 \cdot 19 \cdot 18} = \frac{1}{285}
\end{aligned}$$

So, the expected number of defects in the sample is

$$\begin{aligned}
E[X] &= 0 \cdot \mathbf{P}\{X=0\} + 1 \cdot \mathbf{P}\{X=1\} + 2 \cdot \mathbf{P}\{X=2\} + 3 \cdot \mathbf{P}\{X=3\} \\
&= 0 \cdot \frac{28}{57} + 1 \cdot \frac{8}{19} + 2 \cdot \frac{8}{95} + 3 \cdot \frac{1}{285} \\
&= \frac{3}{5}
\end{aligned}$$

35. Let X be the amount won.

$$\begin{aligned}
\mathbf{P}\{X=-1.00\} &= \frac{2 \cdot 5^2}{10 \cdot 9} = \frac{5}{9} \\
\mathbf{P}\{X=1.10\} &= 1 - \mathbf{P}\{X=1.10\} = \frac{4}{9}
\end{aligned}$$

(a). The expected value of the amount won is $-\frac{1}{15} \approx 0.0667$:

$$E[X] = (-1.00) \cdot \frac{5}{9} + (1.10) \cdot \frac{4}{9} = -\frac{1}{15} \approx 0.0667$$

(b). The variance of the amount won is $\frac{49}{45} \approx 1.0889$:

$$\begin{aligned}
E[X^2] &= (-1.00)^2 \cdot \frac{5}{9} + (1.10)^2 \cdot \frac{4}{9} = \frac{82}{75} \\
E[X]^2 &= \frac{1}{225} \\
Var(x) &= \frac{82}{75} - \frac{1}{225} = \frac{49}{45}
\end{aligned}$$

38. The solutions are explained below.

$$\begin{aligned}
(a) \quad E[X+2] &= 14 \\
(b) \quad Var(4+3X) &= 45.
\end{aligned}$$

(a). We are given $E[X]=1$ and $Var(X)=5$. Use identities to compute:

$$\begin{aligned}
Var(X+2) &= Var(X) = 5 \quad \text{p. 150} \\
E[X+2] &= E[X] + 2 = 3 \quad \text{p. 148}
\end{aligned}$$

Now use the definition of variance

$$\begin{aligned}
Var(X+2) &= E[(X+2)^2] - E[X+2]^2 \qquad \text{so} \\
E[(X+2)^2] &= V(X+2) + E[X]^2 = 14
\end{aligned}$$

(b). By the identity on page 150,

$$Var(4 + 3X) = 9 \cdot Var(X) = 45.$$