MATH 425
HOMEWORK 3
Winter, 2009

5. Consider the following two events:

- E: First two balls selected are white,
- F: Last two balls selected are black.

We want $\mathbf{P}(E \cap F)$:

$$\mathbf{P}(E \cap F) = \mathbf{P}(F|E) \cdot \mathbf{P}(E).$$

The right-side is easy to compute:

$$\begin{aligned} \mathbf{P}(E) &= \frac{6 \cdot 5}{15 \cdot 14} \\ \mathbf{P}(F|E) &= \frac{9 \cdot 8}{13 \cdot 12} \end{aligned}$$

So,

$$\mathbf{P}(E \cap F) = \frac{9 \cdot 8 \cdot 6 \cdot 5}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{6}{91} \approx 0.0659$$

12. Consider the following events:

- E_i: She passes ith test ($i = 1, 2, 3$).
- F: She does not pass all three exams.

Note that $F = (E_1 \cap E_2 \cap E_3)^c$.

(a). We want to compute

$$\mathbf{P}(E_1 \cap E_2 \cap E_3) = \mathbf{P}(E_1) \cdot \mathbf{P}(E_2 \,|\, E_1) \cdot \mathbf{P}(E_3 \,|\, E_1 \cap E_2).$$

The probabilities on the right-side are given:

$$\mathbf{P}(E_1) = 0.9 \quad \mathbf{P}(E_2 \,|\, E_1) = 0.8 \quad \mathbf{P}(E_3 \,|\, E_1 \cap E_2) = 0.7.$$

So,

$$\mathbf{P}(E_1 \cap E_2 \cap E_3) = 0.9 \cdot 0.8 \cdot 0.7 = 0.504.$$

(b). We want to compute

$$\begin{aligned}\mathbf{P}(E_1 \cap E_2^c \,|\, F) &= \frac{\mathbf{P}(F \cap E_1 \cap E_2^c)}{\mathbf{P}(F)} \\ &= \frac{\mathbf{P}(F \,|\, E_1 \cap E_2^c) \cdot \mathbf{P}(E_1 \cap E_2^c)}{\mathbf{P}(F)}.\end{aligned}$$

(We want to know the probability she *fails* the second exam, which requires she passes the first exam.)

The right-side can be compute as follows.

$$\mathbf{P}(F \,|\, E_2^c) = 1 \qquad \mathbf{P}(F) = 1 - 0.504 = 0.496 \;\; \text{(from part a)}\;,$$

and

$$\begin{aligned}\mathbf{P}(E_1 \cap E_2^c) &= \mathbf{P}(E_2^c \,|\, E_1) \cdot \mathbf{P}(E_1) \\ &= 0.2 \cdot 0.9 = 0.18.\end{aligned}$$

So,

$$\mathbf{P}(E_1 \cap E_2^c \,|\, F) = \frac{0.18}{0.496} = 0.3629.$$

19. Consider the events

- M: a male in the class,
- F: a female in the class,
- Q: a person who has quit smoking one year.

Note that we are given

$$\mathbf{P}(M) = 0.62 \quad \mathbf{P}(F) = 0.38 \quad \mathbf{P}(Q \,|\, M) = 0.37 \quad \mathbf{P}(Q \,|\, F) = 0.48.$$

(a). We want to compute

$$\begin{aligned}\mathbf{P}(F \,|\, Q) &= \frac{\mathbf{P}(F \cap Q)}{\mathbf{P}(Q)} \\ &= \frac{\mathbf{P}(Q \,|\, F) \cdot \mathbf{P}(F)}{\mathbf{P}(Q)} \\ &= \frac{0.48 \cdot 0.38}{0.4118} \approx 0.4429.\end{aligned}$$

(See part (b) for $\mathbf{P}(Q)$.)

(b).

$$\begin{aligned}\mathbf{P}(Q) &= \mathbf{P}(Q \,|\, F) \cdot \mathbf{P}(F) + \mathbf{P}(Q \,|\, M) \cdot \mathbf{P}(M) \\ &= 0.48 \cdot 0.38 + 0.37 \cdot 0.62 = 0.4118.\end{aligned}$$

26. Consider the events

- M: a male in the class,
- F: a female in the class,
- C: a person who is colorblind.

Note that we are given

$$\mathbf{P}(C \,|\, M) = 0.05 \quad \mathbf{P}(C \,|\, F) = 0.0025.$$

(a). We are given $\mathbf{P}(M) = \mathbf{P}(F)$. We want to compute

$$\begin{aligned}\mathbf{P}(M \,|\, C) &= \frac{\mathbf{P}(M \cap C)}{\mathbf{P}(C)} \\ &= \frac{\mathbf{P}(C \,|\, M) \cdot \mathbf{P}(M)}{\mathbf{P}(C)}.\end{aligned}$$

We need (see Formula 3.3.1, p. 72)

$$\begin{aligned}\mathbf{P}(C) &= \mathbf{P}(C \,|\, M) \cdot \mathbf{P}(M) + \mathbf{P}(C \,|\, F) \cdot \mathbf{P}(F) \\ &= 0.05 \cdot 0.5 + 0.0025 \cdot 0.5 = 0.02625.\end{aligned}$$

So,

$$\mathbf{P}(M \,|\, C) = \frac{0.05 \cdot 0.5}{0.02625} = \frac{20}{21} \approx 0.9524.$$

(b). Suppose $\mathbf{P}(M) = 2\mathbf{P}(F)$, so $\mathbf{P}(M) = \frac{2}{3}$. So,

$$\begin{aligned}\mathbf{P}(C) &= \mathbf{P}(C \,|\, M) \cdot \mathbf{P}(M) + \mathbf{P}(C \,|\, F) \cdot \mathbf{P}(F) \\ &= 0.05 \cdot \frac{2}{3} + 0.0025 \cdot \frac{1}{3} \approx 0.0342.\end{aligned}$$

and

$$\mathbf{P}(M \,|\, C) \;=\; \frac{0.05 \cdot \frac{2}{3}}{0.0342} = \frac{40}{41} \approx 0.9756.$$

29. Consider the events

- U_i: i balls selected in the first three are unused ($i = 0, 1, 2, 3$),
- B_i: the ith ball in the second selection is unused ($i = 1, 2, 3$),
- V: all balls in the second selection are unused. Note that

$$V = B_1 \cap B_2 \cap B_3$$

We want to compute $\mathbf{P}(V)$. Since the events U_i are mutually exclusive and exhaustive. So, by equation 3.3.4 (p. 81)

$$\mathbf{P}(V) = \sum_{i=0}^{3} \mathbf{P}(V \,|\, U_i)\mathbf{P}(U_i).$$

We can apply the multiplication rule (p. 71)

$$\begin{aligned}
\mathbf{P}(V \,|\, U_i) &= \mathbf{P}(B_1 \cap B_2 \cap B_3 \,|\, U_i) \\
&= \frac{\mathbf{P}(B_1 \cap B_2 \cap B_3 \cap U_i)}{\mathbf{P}(U_i)} \\
&= \mathbf{P}(C_1 \,|\, U_i) \cdot \mathbf{P}(C_2 \,|\, U_i \cap C_1) \cdot \mathbf{P}(C_3 \,|\, U_i \cap C_1 \cap C_2)
\end{aligned}$$

We can compute the required probabilites. First,

$$\begin{aligned}
\mathbf{P}(U_0) &= \binom{3}{0} \frac{6 \cdot 5 \cdot 4}{15 \cdot 14 \cdot 13} \\
\mathbf{P}(U_1) &= \binom{3}{1} \frac{9 \cdot 6 \cdot 5}{15 \cdot 14 \cdot 13} \\
\mathbf{P}(U_2) &= \binom{3}{2} \frac{9 \cdot 8 \cdot 6}{15 \cdot 14 \cdot 13} \\
\mathbf{P}(U_3) &= \binom{3}{3} \frac{9 \cdot 8 \cdot 7}{15 \cdot 14 \cdot 13}
\end{aligned}$$

Reason: Choose i balls out of 3 selections to be white, the rest black, and compute the probabilities by conditionalizing with the multiplication rule. For example, to compute the selection of BWW in the first selection:

$$\begin{aligned}
\mathbf{P}(BWW) &= \mathbf{P}(B) \cdot \mathbf{P}(W \mid B) \cdot \mathbf{P}(W \mid BW) \\
&= \frac{6 \cdot 9 \cdot 8}{15 \cdot 14 \cdot 13}.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{P}(V \mid U_0) &= \frac{9 \cdot 8 \cdot 7}{15 \cdot 14 \cdot 13} \\
\mathbf{P}(V \mid U_1) &= \frac{8 \cdot 7 \cdot 6}{15 \cdot 14 \cdot 13} \\
\mathbf{P}(V \mid U_2) &= \frac{7 \cdot 6 \cdot 5}{15 \cdot 14 \cdot 13} \\
\mathbf{P}(V \mid U_3) &= \frac{6 \cdot 5 \cdot 4}{15 \cdot 14 \cdot 13}
\end{aligned}$$

We can compute the desired probabilities by plugging in the values into the first equation:

$$\mathbf{P}(V) = \frac{528}{5915} \approx 0.0893.$$

36. Consider the events

- A, B, C: person is employed at store A, B, C (respectively),
- F: person is a female,
- R: employee resigns.

We are given the following probabilities

$$\begin{array}{lll}
\mathbf{P}(A) = \dfrac{50}{225} & \mathbf{P}(B) = \dfrac{75}{225} & \mathbf{P}(C) = \dfrac{100}{225} \\
\mathbf{P}(F \mid A) = 0.5 & \mathbf{P}(F \mid B) = 0.6 & \mathbf{P}(F \mid C) = 0.7
\end{array}$$

We are also told the all employees (from all stores combined) are equally likely to resign, even when we take into account gender. This means that A, B, C are conditionally independent of R given F; that is,

$$\mathbf{P}(C \mid F \cap R) = \mathbf{P}(C \mid F).$$

We want to compute $\mathbf{P}(C \mid F \cap R)$, so it is sufficient to compute

$$\begin{aligned}
\mathbf{P}(C \mid F) &= \frac{\mathbf{P}(F \mid C) \cdot \mathbf{P}(C)}{\mathbf{P}(F)} \\
&= \frac{\mathbf{P}(F \mid C) \cdot \mathbf{P}(C)}{\mathbf{P}(F \mid A) \cdot \mathbf{P}(A) + \mathbf{P}(F \mid B) \cdot \mathbf{P}(B) + \mathbf{P}(F \mid C) \cdot \mathbf{P}(C)} \\
&= \frac{0.7 \cdot \frac{100}{225}}{0.5 \cdot \frac{50}{225} + 0.6 \cdot \frac{75}{225} 0.7 \cdot \frac{100}{225}} \\
&= \frac{1}{2}.
\end{aligned}$$

(Where we used Baye's Theorem, Proposition 3.3.1.)

47. Consider the events

- W: All balls selected are white.
- B_i: i balls are selected ($i = 1, 2, 3, 4, 5, 6$).

We are given the probabilities $\mathbf{P}(B_i) = \frac{1}{6}$. First, compute

$$\begin{aligned}
\mathbf{P}(W \mid B_i) &= \frac{\binom{5}{i}}{\binom{15}{i}} \quad 1 \le i \le 5 \\
\mathbf{P}(W \mid B_6) &= 0.
\end{aligned}$$

In the first case we need to select i white balls out of 5, when there are $\binom{15}{i}$ ways to choose i balls out of 15 in the urn. It is impossible to draw 6 white balls.

(a) We want to compute

$$\begin{aligned}
\mathbf{P}(W) &= \sum_{i=1}^{6} \mathbf{P}(W \mid B_i)\mathbf{P}(B_i) \\
&= \frac{1}{6} \sum_{i=1}^{5} \mathbf{P}(W \mid B_i) \\
&= \frac{1}{6} \cdot \left[\frac{1}{3} + \frac{2}{21} + \frac{2}{91} + \frac{1}{273} + \frac{1}{3003}\right] \\
&= \frac{5}{66} \approx 0.07576.
\end{aligned}$$

(b). We want to compute

$$\begin{aligned}\mathbf{P}(B_3 \,|\, W) &= \frac{\mathbf{P}(W \,|\, B_3) \cdot \mathbf{P}(B_3)}{\mathbf{P}(W)} \\ &= \frac{\frac{1}{6} \cdot \frac{2}{91}}{\frac{5}{66}} \\ &= \frac{22}{455} \approx 0.04385.\end{aligned}$$

55. Consider the events

- G: person is a girl.
- F: person is a freshman.

Since we want gender and class to be independent, we need

$$\mathbf{P}(G) = \mathbf{P}(G \,|\, F) = 0.6$$

Let x be the number of sophomore girls we need, then

$$\mathbf{P}(G) = 0.6 = \frac{6 + x}{4 + 6 + 6 + x},$$

so $x = 9$.

60. I will write B for the dominant brown-eyed gene and b for the recessive blue-eyed gene. I am writing BB, Bb, bB, bb for the possible gene pairs. We know that both parents are Bb or bB (there is no difference) – the reason is that both have brown-eyes and they must carry a b gene, since Smith's sister is bb (she has blue eyes).

Smith (who has brown eyes) is BB, Bb or bB. Consider the events

- E_{BB}, E_{Bb}, E_{bB}: Smith has gene pattern BB, Bb, bB (respectively).

We know that each of these events is equally likely, and together exhaustive, so

$$\mathbf{P}(E_{BB}) = \mathbf{P}(E_{Bb}) = \mathbf{P}(E_{bB}) = \frac{1}{3}.$$

(a). Let $E_b = E_{Bb} \cup E_{bB}$ be the event that Smith has a b gene. Then by equation 3.3.4 (p. 81)

$$\begin{aligned}\mathbf{P}(E_b) &= \mathbf{P}(E_b \mid E_{BB}) \cdot \mathbf{P}(E_{BB}) + \mathbf{P}(E_b \mid E_{Bb}) \cdot \mathbf{P}(E_{Bb}) + \mathbf{P}(E_b \mid E_{bB}) \cdot \mathbf{P}(E_{bB}) \\ &= 0 + \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

(b). Since Smith's wife has blue eyes, her gene pattern is bb. Let

- C: The first child has blue eyes.

Any child of Smith will have blue eyes provided Smith gives it a b gene. So,

$$\mathbf{P}(C \mid E_b) = \frac{1}{2} \qquad \mathbf{P}(C \mid E_b^c) = 0$$

The probability that Smith's first child has blue eyes is

$$\begin{aligned}\mathbf{P}(C) &= \mathbf{P}(C \mid E_b) \cdot \mathbf{P}(E_b) + \mathbf{P}(C \mid E_b^c) \cdot \mathbf{P}(E_b^c) \\ &= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.\end{aligned}$$

(c). Consider the events

- C_i: Child i has brown eyes ($i = 1, 2$).

Now we have for each $i = 1, 2$,

$$\mathbf{P}(C_i \mid E_b) = \frac{1}{2} \qquad \mathbf{P}(C_i \mid E_b^c) = 1$$

We want to compute

$$\begin{aligned}\mathbf{P}(C_2 \mid C_1) &= \mathbf{P}(C_2 \cap E_b \mid C_1) + \mathbf{P}(C_2 \cap E_b^c \mid C_1) \\ &= \mathbf{P}(C_2 \mid C_1 \cap E_b) \cdot \mathbf{P}(E_b \mid C_1) + \mathbf{P}(C_2 \mid C_1 \cap E_b^c) \cdot \mathbf{P}(E_b^c \mid C_1)\end{aligned}$$

Notice that

$$\begin{aligned}\mathbf{P}(C_2 \mid C_1 \cap E_b) &= \mathbf{P}(C_2 \mid E_b) = \frac{1}{2} \\ \mathbf{P}(C_2 \mid C_1 \cap E_b^c) &= \mathbf{P}(C_2 \mid E_b^c) = 1\end{aligned}$$

The second equality is clear since if E_b is true, then there is only one B gene for Smith to pass on, and if E_b^c is true, then Smith has two B genes to pass on. The first equality is because C_1 and C_2 are conditionally independent given E_b (see page 108).

We plug in these equivalences:

$$\begin{aligned}
\mathbf{P}(C_2 \mid C_1) &= \frac{1}{2} \cdot \mathbf{P}(E_b \mid C_1) + \mathbf{P}(E_b^c \mid C_1) \\
&= \frac{1}{2} \cdot \frac{\mathbf{P}(C_1 \mid E_b) \cdot \mathbf{P}(E_b)}{\mathbf{P}(C_1)} + \frac{\mathbf{P}(C_1 \mid E_b^c) \cdot \mathbf{P}(E_b^c)}{\mathbf{P}(C_1)} \\
&= \frac{1}{2} \cdot \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{2}{3}} + \frac{1 \cdot \frac{1}{3}}{\frac{2}{3}} \\
&= \frac{3}{4}.
\end{aligned}$$

70. Consider the events

- Q: the queen is a carrier,
- E_i: the ith prince has hemophilia ($i = 1, 2, 3, 4$),
- F: None of the first three princes has hemophilia. So,

$$F = E_1^c \cap E_2^c \cap E_3^c.$$

We are given the following probabilities

$$\mathbf{P}(Q) = \frac{1}{2} \qquad \mathbf{P}(E_i^c \mid Q) = \frac{1}{2} \quad \mathbf{P}(E_i^c \mid Q^c) = 1.$$

But we also have that each of the events E_i and E_i^c are conditionally independent given the queen is a carrier, Q. So,

$$\mathbf{P}(F \mid Q) = \frac{1}{2}^3 = \frac{1}{8} \qquad \mathbf{P}(F \mid Q^c) = 1.$$

(a). We want to compute

$$\begin{aligned}
\mathbf{P}(Q \mid F) &= \frac{\mathbf{P}(F \mid Q) \cdot \mathbf{P}(Q)}{\mathbf{P}(F)} \\
&= \frac{\mathbf{P}(F \mid Q) \cdot \mathbf{P}(Q)}{\mathbf{P}(F \mid Q) \cdot \mathbf{P}(Q) + \mathbf{P}(F \mid Q^c) \cdot \mathbf{P}(Q^c)} \\
&= \frac{\frac{1}{8} \cdot \frac{1}{2}}{\frac{1}{8} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} \\
&= \frac{1}{9}.
\end{aligned}$$

(b). Suppose there is a fourth prince. By independence,

$$\mathbf{P}(E_4 \cap F \mid Q) = \frac{1}{16} \qquad \mathbf{P}(E_4 \cap F \mid Q^c) = 0.$$

We want to compute

$$\begin{aligned}
\mathbf{P}(E_4 \mid F) &= \frac{\mathbf{P}(E_4 \cap F)}{\mathbf{P}(F)} \\
&= \frac{\mathbf{P}(E_4 \cap F \mid Q) \cdot \mathbf{P}(Q) + \mathbf{P}(E_4 \cap F \mid Q^c) \cdot \mathbf{P}(Q^c)}{\mathbf{P}(F \mid Q) \cdot \mathbf{P}(Q) + \mathbf{P}(F \mid Q^c) \cdot \mathbf{P}(Q^c)} \\
&= \frac{\frac{1}{16} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}}{\frac{1}{8} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} \\
&= \frac{1}{18}.
\end{aligned}$$