MATH 425
HOMEWORK 2
Winter, 2009

7. Outcomes in the space are sequences of length 15, where each term is a pair (x, y) determined as follows:

- $x \in \{0, 1\}$, where $x = 0$ means 'x is a blue-collar worker' and $x = 1$ means 'x is a white-collar worker' ,
- $y \in \{0, 1, 2\}$, where $y = 0$ means '$y = 0$ is a Democrat', $y = 1$ means 'y is an Independent', and $y = 2$ means 'y is a Republican'.

(a) The sample space has $\big(2 \cdot 3\big)^{15} = 470,184,984,576$ outcomes. (There are $2 \cdot 3$ pairs (x, y), and a sequence of 15 pairs.)

(b) There are $\big(2 \cdot 3\big)^{15} - \big(1 \cdot 3\big)^{15} = 470,170,635,669$ outcomes with at least one blue-collar worker. Here is how I computed this:

- Let S be the sample space of all outcomes, and E be the outcomes in which at least one worker is blue-collar. Then, E^c is the outcomes where no worker is blue-collar, so $x = 0$ for all pairs in these outcomes. This is easy to count:

$$|E^c| = 3^{15}.$$

Then,

$$|E| = |S| - |E^c| = \big(2 \cdot 3\big)^{15} - 3^{15}.$$

(c) There are $\big(2 \cdot 2\big)^{15} = 1,073,741,824$ outcomes in which none of the members considers themselves an Independent.

8. A and B are mutually exclusive events for which $\mathbf{P}(A) = 0.3$ and $\mathbf{P}(B) = 0.5$.

(a) $\mathbf{P}(A \cup B) = 0.3 + 0.5 = 0.8$,

(b) $\mathbf{P}(A \cap B^c) = 0.3$. The reason: since $A \cap B = \emptyset$,

$$A = (A \cap B) \cup (A \cap B^c) = \emptyset \cup (A \cap B^c),$$

so that $A \cap B^c = A$.

(c) $\mathbf{P}(A \cap B) = 0$ (since $A \cap B = \emptyset$, by hypothesis).

11. We have the following events:

- A: the event of American males who smoke cigarettes,
- B: the event of American males who smoke cigars.

We are given the following probabilities:

$$\mathbf{P}(A) = 0.28 \quad \mathbf{P}(B) = 0.07 \quad \mathbf{P}(A \cap B) = 0.05$$

(a) $\mathbf{P}\big((A \cup B)^c\big) = 0.7$. The reason: we have by Proposition 4.1,

$$\mathbf{P}\big((A \cup B)^c\big) = 1 - \mathbf{P}\big(A \cup B\big),$$

and we can compute the right-side using Proposition 4.3:

$$\mathbf{P}\big(A \cup B\big) = \mathbf{P}\big(A\big) + \mathbf{P}\big(B\big) - \mathbf{P}\big(A \cup B\big) = 0.28 + 0.07 - 0.05 = 0.3.$$

(b) $\mathbf{P}(B \cap A^c) = 0.02$. The reason: since

$$B = (B \cap A) \cup (B \cap A^c),$$

and the two events on the right-side are mutually exclusive. So, by Axiom 3:

$$\mathbf{P}(B \cap A^c) = \mathbf{P}(B) - \mathbf{P}(B \cap A) = 0.07 - 0.05 = 0.02.$$

12. We have the following events:

- A: students who speak Spanish,
- F: students who speak French,
- G: students who speak German

There are 100 students in total, and so we can compute the following probabilities:

$$\begin{aligned}
&\mathbf{P}(A) = 0.28 \quad \mathbf{P}(F) = 0.26 \quad \mathbf{P}(G) = 0.16 \\
&\mathbf{P}(A \cap F) = 0.12 \quad \mathbf{P}(A \cap G) = 0.04 \quad \mathbf{P}(F \cap G) = 0.06 \\
&\mathbf{P}(A \cap F \cap G) = 0.02.
\end{aligned}$$

(a) $\mathbf{P}\big((A \cup F \cup G)^c\big) = 0.5$. The reason: by Proposition 4.1,

$$\mathbf{P}\big((A \cup F \cup G)^c\big) = 1 - \mathbf{P}\big(A \cup F \cup G\big),$$

and we can compute the right-side by the Inclusion-Exclusion Identity (Proposition 4.4) for three events:

$$\begin{aligned}
\mathbf{P}\big(A \cup F \cup G\big) &= \mathbf{P}(A) + \mathbf{P}(F) + \mathbf{P}(G) - \big(\mathbf{P}(A \cap F) + \mathbf{P}(A \cap G) + \mathbf{P}(F \cap G)\big) \\
&\qquad + \mathbf{P}(A \cap F \cap G) \\
&= 0.28 + 0.26 + 0.16 - (0.12 + 0.04 + 0.06) + 0.02 \\
&= 0.5
\end{aligned}$$

(b) The probability of a student taking exactly one class is 0.32.
The reason: we can express the following events,

$A \cap F^c \cap G^c$	students taking exactly one Spanish class,
$F \cap A^c \cap G^c$	students taking exactly one French class,
$G \cap A^c \cap F^c$	students taking exactly one German class.

Note that these events are mutually exclusive. Let X be the event that a student is taking exactly one language class, then by Axiom 3:

$$\mathbf{P}(X) = \mathbf{P}(A \cap F^c \cap G^c) + \mathbf{P}(F \cap A^c \cap G^c) + \mathbf{P}(G \cap A^c \cap F^c).$$

Here is how we can compute the first term on the right-side:

$$\begin{aligned}
A &= \big(A \cap (F \cup G)^c\big) \cup \big(A \cap (F \cup G)\big) \\
&= \big(A \cap F^c \cap G^c\big) \cup \big(A \cap (F \cup G)\big) \quad \text{DeMorgan} \\
&= \big(A \cap F^c \cap G^c\big) \cup \big((A \cap F) \cup (A \cap G)\big) \quad \text{Distributivity;}
\end{aligned}$$

so,

$$\mathbf{P}\big(A \cap F^c \cap G^c\big) = \mathbf{P}(A) - \mathbf{P}\big((A \cap F) \cup (A \cap G)\big).$$

We can compute the right-side using Proposition 4.2

$$\mathbf{P}\big((A \cap F) \cup (A \cap G)\big) = \mathbf{P}(A \cap F) + \mathbf{P}(A \cap G) - \mathbf{P}(A \cap F \cap G),$$

and putting this together:

$$\mathbf{P}\big(A \cap F^c \cap G^c\big) = \mathbf{P}(A) - \big(\mathbf{P}(A \cap F) + \mathbf{P}(A \cap G)\big) + \mathbf{P}(A \cap F \cap G) = 0.26 - 0.12 = 0.14$$

We compute all three events in the same way:

$$\mathbf{P}\big(A \cap F^c \cap G^c\big) = 0.14 \quad \mathbf{P}\big(F \cap A^c \cap G^c\big) = 0.10 \quad \mathbf{P}\big(G \cap A^c \cap F^c\big) = 0.08.$$

(c) The probability that at least one of two students chosen at random is taking a language class is $\frac{149}{198} = 0.7525$. The reason: we know from (a) that there are 50 students taking a language class and 50 students not taking a language class. There are $\binom{100}{2}$ ways of choosing two students, and these are all equally likely. There are also $\binom{50}{2}$ ways of choosing two students who **do not take any language class**. So, the probability of choosing two student who take no language class is

$$\frac{\binom{50}{2}}{\binom{100}{2}} = \frac{49}{198},$$

and the probability that at least one student takes a language class is then

$$1 - \frac{49}{198} = \frac{149}{198}$$

15. All $\binom{52}{5}$ poker hands are equally possible.

(a) The probability of a flush

$$\frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = 0.002.$$

Reason: choose a suit (4 choices), then choose 5 cards from that suit.

(b) The probability of one pair is

$$\frac{13 \cdot \binom{4}{2} \cdot \frac{48 \cdot 44 \cdot 40}{3!}}{\binom{52}{5}} = 0.4226.$$

Reason: choose a face value for the pair (14 choices), choose 2 out of 4 possible cards of that face value ($\binom{4}{2}$ choices), then choose the remaining cards – each with different face values ($\frac{48 \cdot 44 \cdot 40}{3!}$ choices).

(c) The probability of two pairs is

$$\frac{\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44}{\binom{52}{5}} = 0.0475.$$

Reason: choose two face values ($\binom{13}{2}$ choices), choose 2 out of 4 possible cards for each of the two face values ($\binom{4}{2} \cdot \binom{4}{2}$), then choose the final card to avoid the previous two face values (44 choices).

(d) The probability of three pairs is

$$\frac{13 \cdot \binom{4}{3} \cdot \frac{48 \cdot 44}{2!}}{\binom{52}{5}} = 0.0211.$$

Reason: Choose one face value (13 choices), choose 3 out of 4 possible cards of that face value ($\binom{4}{3}$), then choose the final two cards – each with different face values ($\frac{48 \cdot 44}{2!}$ choices).

(e) The probability of three pairs is

$$\frac{13 \cdot 48}{\binom{52}{5}} = 0.00024.$$

Reason: Choose one face value (13 choices), then choose the last card with a different face value (48 choices).

17. The probability that no row or file on a chess board contains more than one rook is

$$\frac{\frac{64 \cdot 49 \cdot 36 \cdot 25 \cdot 16 \cdot 9 \cdot 4 \cdot 1}{8!}}{\binom{64}{8}} = 9.10947 \times 10^{-6}.$$

The reason: There are $\binom{64}{8}$ possible placements. Now, for each piece we choose 1 row and 1 file. For the first piece we can choose any of 8 rows and 8 files, or $8 \cdot 8 = 64$ possible placements; for the second piece we can choose any of 7 row and 7 files, or $7 \cdot 7 = 49$ possible placements avoiding the row and file of the previous piece; and so on. We can continue this process to get the total number of choices for placing 8 pieces:

$$\frac{64 \cdot 49 \cdot 36 \cdot 25 \cdot 16 \cdot 9 \cdot 4 \cdot 1}{8!},$$

(we divide by 8! since the order of placing the pieces does not matter.)

27. The probability that A selects the first red ball is $\frac{7}{12} = 0.58333$.

Reason: We are interested in the following mutually exclusive events

- E_i: A draws the first red ball on the ith pick (where $i = 1, 3, 5, 7$)

So, the event of A drawing the first ball is

$$E_1 \cup E_3 \cup E_5 \cup E_7$$

and the probability of A choosing the first ball is then

$$\mathbf{P}(E_1) + \mathbf{P}(E_3) + \mathbf{P}(E_5) + \mathbf{P}(E_7).$$

We will number each ball $1, 2, \ldots, 10$, and the sample space S will consist of all permutations of these numbers: $10!$.

Compute $\mathbf{P}(E_1)$. A can choose any 1 of three red balls, and the remaining 9 choices can be selected in any order:

$$\mathbf{P}E_i = \frac{3 \cdot 9!}{10!} = \frac{3}{10}.$$

Compute $\mathbf{P}(E_i)$, for $i > 1$. The event E_i requires that A and B choose $i-1$ black balls on the first $i-1$ selections, and A chooses 1 red ball on the ith selection. For the remaining $10-i$ balls any order of selection is possible. (Only the first i balls are relevant to determing whether E_i occurred.) The number of possible outcomes is

$$\frac{7! \cdot 3 \cdot (10-i)!}{(7-i+1)!} \qquad \text{since } \frac{7!}{(7-i+1)!} = 7 \cdots (7-i+2)$$

So, when $i > 1$:

$$\mathbf{P}(E_i) = \frac{7! \cdot 3 \cdot (10-i)!}{(7-i)!10!}.$$

Here are the probabilities for these events:

$$\mathbf{P}(E_1) = \frac{3}{10} \quad \mathbf{P}(E_3) = \frac{7}{40} \quad \mathbf{P}(E_5) = \frac{1}{12} \quad \mathbf{P}(E_7) = \frac{1}{40}.$$

46. The fewest number of people in a room before there is a probability of at least $\frac{1}{2}$ that two will share the same birth month is 5.

Suppose k people are chosen at random. Let E_k be the event that at least two people have the same birth month. Our sample space for this experiment are sequences

$$(x_1, x_2, \ldots, x_{12}) \qquad \text{where } 1 \leq x_i \leq 12 \text{ for each } x_i$$

(where x_i records the birth month of the ith person). This sample space S_k has 12^k outcomes, and each is equally likely from our assumption that each month is equally likely to be a birth month.

If $k > 12$, then there must be two people with the same birth month. So, suppose $k \leq 12$. It is easier to compute E_k^c, the event that no two people chosen have the same birth month:

$$\mathbf{P}(E_k^c) = \frac{|E_k^c|}{12^k} = \frac{12 \cdot 11 \cdots (12 - k + 1)}{12^k}$$

and from this we can compute $\mathbf{P}(E_k)$:

$$\mathbf{P}(E_k) = 1 - \mathbf{P}(E_k^c).$$

We want the smallest k with $\mathbf{P}(E_k) \geq \frac{1}{2}$. Here are the values of E_k for the $k = 2, 3, 4, 5$:

$$\mathbf{P}(E_2) = 0.0833 \quad \mathbf{P}(E_3) = 0.2361 \quad \mathbf{P}(E_4) = 0.4271 \quad \mathbf{P}(E_5) = 0.6181.$$

53. The probability of none of 4 couples sitting in a row sit together is $\frac{12}{35}$.

There are 8! ways of arranging eight people in a row. We will consider the following events (for $i = 1, 2, 3, 4$):

- E_i: Couple i are sitting together.

Then the event $\big(E_1 \cup E_2 \cup E_3 \cup E_4\big)^c$ is the event that no couple sits together. To compute this probability, we use the fact that

$$\mathbf{P}\big((E_1 \cup E_2 \cup E_3 \cup E_4)^c\big) = 1 - \mathbf{P}\big(E_1 \cup E_2 \cup E_3 \cup E_4\big).$$

Since the events E_i (for $i = 1, 2, 3, 4$) are NOT mutually exclusive, we need to apply Proposition 4.4 with $n = 4$. Here are the probabilities we need:

$$\begin{aligned}
\mathbf{P}(E_i) &= \frac{7! \cdot 2!}{8!} = \frac{1}{4} & i = 1, 2, 3, 4 \\
\mathbf{P}(E_i \cap E_j) &= \frac{6! \cdot (2!)^2}{8!} = \frac{1}{14} & 1 \leq i < j \leq 4 \\
\mathbf{P}(E_i \cap E_j \cap E_k) &= \frac{5! \cdot (2!)^3}{8!} = \frac{1}{42} & 1 \leq i < j < k \leq 4 \\
\mathbf{P}(E_1 \cap E_2 \cap E_3 \cap E_4) &= \frac{4!(2!)^4}{8!} = \frac{1}{105}
\end{aligned}$$

For example, to compute $\mathbf{P}(E_i \cap E_j)$, treat the ith and jth couples as one person (since they must sit together) and consider all arrangements of 6 persons: 6!. Once an arrangement is selected, we must consider the arrangement of each pair of persons in each of the couples sitting together: $2! \cdot 2!$.

Now, we can use Proposition 4.4:

$$\begin{aligned}
\mathbf{P}\big(E_1 \cup E_2 \cup E_3 \cup E_4\big) &= \binom{4}{1} \Pr(E_i) - \binom{4}{2}\mathbf{P}(E_i \cap E_j) + \binom{4}{3}\mathbf{P}(E_i \cap E_j \cap E_k) - \\
&\quad \binom{4}{4}\mathbf{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \\
&= 4(\frac{1}{4}) - 6(\frac{1}{14}) + 4(\frac{1}{42}) - \frac{1}{105} = \frac{23}{35}.
\end{aligned}$$

The first equation is obtained as follows. For the coefficient in front of $\mathbf{P}(E_i \cap E_j)$: there are $\binom{4}{2}$ ways of choosing $1 \leq i, j \leq 4$ and each of the probabilities $\mathbf{P}(E_i \cap E_j)$ are the same. The rest of the coefficients are obtained similarly.

So, the probability we want can now be computed:

$$\mathbf{P}\big((E_1 \cup E_2 \cup E_3 \cup E_4)^c\big) = 1 - \mathbf{P}\big(E_1 \cup E_2 \cup E_3 \cup E_4\big) = 1 - \frac{23}{35} = \frac{12}{35}.$$