MATH 425
HOMEWORK 10
Winter, 2009

5. If we place the hospital at the origin, then the are it services is the rectangle $[-1.5, 1.5]\times[-1.5, 1.5]$. Let (X, Y) be the coordinates of an accident whose location is randomly distributed in the square. The joint density of (X, Y) is

$$f_{X,Y}(x, y) = \frac{1}{9} \quad \text{if } -1.5 \le x, y \le 1.5.$$

Let $Z = |X| + |Y|$ be the distance an ambulance must travel to get from the hospital to (X, Y). Since $E[Z] = E[|X|] + E[|Y|] = 2E[|X|]$ (X and Y are uniformly distributed in $[-1.5, 1.5]$), it is enough to compute

$$\begin{aligned} E[|X|] &= \int_{-\infty}^{\infty} |x| \cdot f_X(x)\, dx \\ &= \frac{1}{3}\int_{-1.5}^{1.5} |x|\, dx \\ &= \frac{1}{3}\int_{0}^{1.5} x\, dx - \frac{1}{3}\int_{-1.5}^{0} x\, dx \\ &= \frac{2}{3}\int_{0}^{1.5} x\, dx = \left.\frac{x^2}{3}\right|_{x=0}^{1.5} = \frac{3}{4} \end{aligned}$$

Therefore,

$$E[Z] = E[|X|] + E[|Y|] = 2E[|X|] = 2\frac{3}{4} = \frac{3}{2}.$$

7a. Consider the indicator variables for each object i

$$X_i = \begin{cases} 1 & \text{if } A \text{ and } B \text{ chooses } i\ , \\ 0 & \text{otherwise} \end{cases}$$

Since A and B are choosing independently, and each has a $\frac{3}{10}$ chance of choosing object i

$$\mathbf{P}\{X_i = 1\} = \frac{9}{100}.$$

Let X count the number of objects A and B choose together. So, $X = \sum_{i=1}^{10} X_i$, thus the expected number of objects chosen by A and B is

$$E[X] = \sum_{i=1}^{10} E[X_i] = 10 \cdot \frac{9}{100} = \frac{9}{10}.$$

7b. Consider the indicator variables for each object i

$$Y_i = \begin{cases} 1 & \text{if neither } A \text{ nor } B \text{ chooses } i \ , \\ 0 & \text{otherwise} \end{cases}$$

Since A and B are choosing independently, and each has a $\frac{7}{10}$ chance of not choosing object i

$$\mathbf{P}\{Y_i = 1\} = \frac{49}{100}.$$

Let Y count the number of objects neither A nor B chooser. So, $Y = \sum_{i=1}^{10} Y_i$, thus the expected number of objects chosen by A and B is

$$E[Y] = \sum_{i=1}^{10} E[Y_i] = 10 \cdot \frac{49}{100} = 4.9.$$

7c. Consider the indicator variables for each object i

$$Z_i = \begin{cases} 1 & \text{if exactly one of } A \text{ and } B \text{ chooses } i \ , \\ 0 & \text{otherwise} \end{cases}$$

Since A and B are choosing independently, and each has a $\frac{3}{10}$ chance of choosing the object i

$$\begin{aligned} \mathbf{P}\{Z_i = 1\} &= \mathbf{P}\{A \text{ chooses } i, \text{ but not } B\} + \mathbf{P}\{B \text{ chooses } i, \text{ but not } A\} \\ &= \frac{3}{10} \cdot \frac{7}{10} + \frac{7}{10} \cdot \frac{3}{10} \\ &= \frac{42}{100} \end{aligned}$$

Let Z count the number of objects neither A nor B chooser. So, $Z = \sum_{i=1}^{10} Z_i$, thus the expected number of objects chosen by A and B is

$$E[Z] = \sum_{i=1}^{10} E[Z_i] = 10 \cdot \frac{42}{100} = 4.2.$$

9a. Consider the indicator variables U_i for the ith urn

$$U_i = \begin{cases} 1 & \text{if } U_i \text{ is empty} \ , \\ 0 & \text{otherwise} \end{cases}$$

The only balls that could go into urn i are $i, i+1, \dots, n$. When $j \geq i$, the probability that ball j goes into urn i is $\frac{1}{j}$, so the probability that j does NOT go into urn i is $\frac{j-1}{j}$. So,

$$\mathbf{P}\{U_i = 1\} = \prod_{j=i}^{n} \frac{(i-1)\cdot i \cdots (n-1)}{i \cdot (i+1) \cdots n} = \frac{i-1}{n}.$$

Let U count the number of empty urns. Since $U = \sum_{i=1}^{n} U_i$, the expected number of empty urns is

$$\begin{aligned} E[U] &= \sum_{i=1}^{n} E[U_i] \\ &= \sum_{i=1}^{n} \frac{i-1}{n} \\ &= \frac{1}{n} \sum_{i=1} n - 1i \\ &= \frac{1}{n} \cdot \frac{n(n-1)}{2} = \frac{n-1}{2}. \end{aligned}$$

9b. The only way that each urn can contain a ball is for the ith urn to contain only the ith ball. To see this: each urn can only have one ball, if no urn is to be empty. Ball 1 must go in urn 1. Ball 2 can go in urn 1 or urn 2, but since urn 1 has a ball, ball 2 must go into urn 2. Fix $j \leq n$ and assume that ball i is in urn i when $i < j$. Since ball j can only go into urns 1 to j, and each urn already has a ball, except urn j, ball j must go into urn j.

It remains to count the number of possible ways of distributing the balls into urns. Since the ith ball can go into any one of i urns, the total number of possible arrangements is $1 \cdot 2 \cdots n = n!$. So, the probability of no urn being empty is $\frac{1}{n!}$.

11. For each i with $1 \leq i \leq n-1$, consider the following variables indicating a changeover from toss i to $i+1$:

$$X_i = \begin{cases} 1 & \text{if the outcome of toss } i \text{ is different from the outcome of toss } i+1, \\ 0 & \text{otherwise.} \end{cases}$$

To compute the probability of X_i note that if $X_i = 1$ then the outcomes of the ith and $i+1$st trials must be HT or TH. Let $p = \mathbf{P}\{H\}$ and $q = 1-p =$

$\mathbf{P}\{T\}$. Then

$$\mathbf{P}\{X_i = 1\} = \mathbf{P}\{HT\} + \mathbf{P}\{TH\} = 2pq.$$

Let X count the number of changeovers in n tosses. So, $X = \sum_{i=1}^{n-1} X_i$; thus the expected number of changeovers is

$$E[X] = \sum_{i=1}^{n} E[X_i] = (n-1) \cdot 2pq = 2(n-1)pq, \quad \text{equivalently } E[X] = 2(n-1)p(1-p).$$

For example, when $p = \frac{1}{2}$, $E[X] = \frac{n-1}{2}$.

13. Let X count the number of people among 1000 whose age matches the number on their card. Let X_i be the indicator variable that the number on the ith person's card matches their age. Since the ith person is equally likely to get any of the 1000 cards,

$$\mathbf{P}\{X_i = 1\} = \frac{1}{1000}.$$

As $X = \sum_{i=1}^{1000} X_i$,

$$E[X] = \sum_{i=1}^{1000} E[X_i] = 1000 \cdot \frac{1}{1000} = 1.$$

33a. Let $E[X] = 1$ and $Var(X) = 5$.

$$\begin{aligned} E[(2+X)^2] &= E[4 + 2X + X^2] \\ &= E[4] + 4E[X] + E[X^2] \\ &= 4 + 4(1) + \big(Var(X) + E[X]^2\big) \\ &= 8 + (5+1) \\ &= 14. \end{aligned}$$

33b.

$$\begin{aligned} Var(4+3X) &= Var(3X) \\ &= 9Var(X) \\ &= 45. \end{aligned}$$

34a. This problem is based upon Example 2.5n. Let X_i $(i = 1, \ldots, 10)$ be indicator variables for the event that couple i is sat next to each other. So,

$$E[X_i] = \frac{2 \cdot 18!}{19!} = \frac{2}{19}$$

Let X count the number of couples seating next to each other, so $X = \sum_{i=1}^{10} X_i$. Thus,

$$E[X] = \sum_{i=1}^{1} 0E[X_i] = 10 \cdot \frac{2}{19} = \frac{20}{19} \approx 1.053.$$

34b. Let X_i $(i = 1, \ldots, 10)$ be as in 34a. The variance of X_i:

$$Var(X_i) = E[X^2] - \big(E[X]\big)^2 = \frac{2}{19} - \frac{4}{361} = \frac{34}{361}$$

To compute the variance of $X = \sum_{i=1}^{10} X_i$, we must first compute the covariance of X_i and X_j (for $1 \leq i < j \leq 10$). Note that

$$Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_y] \quad \text{and} \quad E[X_i X_j] = \mathbf{P}\{X_i = 1, X_j = 1\}.$$

By example 2.5n,

$$\mathbf{P}\{X_i = 1, X_j = 1\} = \frac{2^2 \cdot 17!}{19!} = \frac{2}{171}.$$

So,

$$Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_y] = \frac{2}{171} - \frac{4}{371} = \frac{2}{3249}.$$

The variance on the number of couples seating next to each other is

$$\begin{aligned} Var(X) &= \sum_{i=1}^{1} 0Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j) \\ &= 10 \cdot \frac{34}{361} + 2 \cdot 45 \cdot \frac{2}{3249} \\ &= \frac{360}{361} \approx 0.997 \end{aligned}$$

35b. Let X be the expected number of cards drawn until 5 spades are drawn. X is a negative hypergeometric random variable (see Example 7.3e in Ross or Lecture 32).

There are 39 nonspades. Number these $1, 2, \ldots, 39$ and let X_i be the indicator variable for the event that the ith nonspade is drawn before five spades have been seen. The probability of this event is

$$\mathbf{P}\{X_i = 1\} = \frac{5}{14}$$

Since $X = 5 + \sum_{i=1}^{3} 9X_i$, the expected number of draws is

$$\begin{aligned} E[X] &= E\big[5 + \sum_{i=1}^{39} X_i\big] \\ &= 5 + \sum_{i=1}^{39} E[X_i] \\ &= 5 + 39 \cdot E[X_1] \\ &= 5 + 39 \cdot \mathbf{P}\{X_i = 1\} \\ &= 5 + 39 \cdot \frac{5}{14} \\ &= \frac{265}{14} \\ &= 18.9286. \end{aligned}$$

36. Consider the indicator variables (for $i = 1, \ldots, n$)

$$X_i = \begin{cases} 1 & \text{if } i\text{th die is `1'}, \\ 0 & \text{otherwise} \end{cases} \qquad Y_i = \begin{cases} 1 & \text{if } i\text{th die is `2'}, \\ 0 & \text{otherwise .} \end{cases}$$

So, $E[X_i] = E[Y_i] = \frac{1}{6}$.

We also need the covariance of X_i and Y_j. If $i = j$, then

$$Cov(X_i, Y_i) = E[X_i Y_i] - E[X_i]E[Y_i] = -\frac{1}{36}.$$

($E[X_iY_i] = 0$, since a die cannot show both '1' and '2'.) If $i \neq j$, then X_i and Y_j are independent, so $Cov(X_i, Y_j) = 0$.

Let X count the number of '1's and Y count the number of '2's on the n die, so

$$X = \sum_{i=1}^{n} X_i \qquad Y = \sum_{i=1}^{n} Y_i$$

The covariance of X and Y is

$$\begin{aligned} Cov(X,Y) &= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, Y_j) \\ &= \sum_{i=1}^{n} Cov(X_i, Y_i) \\ &= -\frac{n}{36} \end{aligned}$$

38. The joint density of X and Y is

$$f(x,y) = \begin{cases} 2e^{-2x}/x & \text{if } 0 \le x < \infty, 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

The covariance is $E[XY] - E[X]E[Y]$. To compute $E[XY]$ use Proposition 7.2.1

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^x (xy) \cdot 2e^{-2x}/x \, dy \, dx \\ &= \int_0^\infty \int_0^x y \cdot 2e^{-2x} \, dy \, dx \\ &= \int_0^\infty x^2 e^{-2x} \, dx \\ &= \frac{1}{4}. \end{aligned}$$

The expectation of X and Y can be computed using the marginal densities

$$\begin{aligned}
E[X] &= \int_0^\infty \int_0^x x \cdot 2e^{-2x}/x \, dy \, dx \\
&= \int_0^\infty \int_0^x \cdot 2e^{-2x} \, dy \, dx \\
&= \int_0^\infty 2xe^{-2x} \, dx \\
&= \frac{1}{2} \\
E[Y] &= \int_0^\infty \int_y^\infty y \cdot 2e^{-2x}/x \, dx \, dy \\
&= \int_0^\infty \int_0^x y \cdot 2e^{-2x}/x \, dy \, dx \\
&= \int_0^\infty xxe^{-2x} \, dx \\
&= \frac{1}{4}.
\end{aligned}$$

So,

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

42a. Let X_i $(i = 1, \ldots, 10)$ be the indicator variable for couple i consists of a man and a woman. Since any person is equally likely to be in couple i,

$$E[X_i] = \frac{10}{19}.$$

(Once you choose the first person, there are 10 out of 19 possibilities of getting a mixed gender couple.)

Let X count the number of couples consisting of a man and a woman. Since $X = \sum_{i=1}^{10} X_i$, so

$$E[X] = \sum_{i=1}^{10} E[X_i] = 10 \cdot \frac{10}{19} = \frac{100}{19} \approx 5.26.$$

To compute the variance of X, note that

$$Var(X_i) = \frac{10}{19} \cdot \frac{9}{19} = \frac{90}{361}.$$

We also need the covariance of X_i and X_j when $i < j$. First

$$E[X_i X_j] = \mathbf{P}\{X_i = 1, X_j = 1\} = \frac{10}{19} \cdot \frac{9}{17} = \frac{90}{323}$$

(Once the first couple is chosen, we must choose another couple out of 9 men and 9 women.) So, the covariance is

$$Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = \frac{90}{323} - \frac{100}{361} = \frac{10}{6137}.$$

Now compute the variance of X

$$\begin{aligned} Var(X) &= \sum_{i=1}^{1} 0Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j) \\ &= 10 \cdot \frac{90}{361} + 2 \cdot 45 \cdot \frac{10}{6137}. \\ &= \frac{16,200}{6137} \approx 2.64 \end{aligned}$$

42b. Let Y_i $(i = 1, \ldots, 10)$ be the indicator variable for couple i consists of a married couple. Since any person is equally likely to be in couple i,

$$E[Y_i] = \frac{1}{19}.$$

(Once you choose the first person, there is only 1 out of 19 possibilities of getting a married couple.)

Let Y count the number of couples consisting of a man and a woman. Since $Y = \sum_{i=1}^{10} Y_i$, so

$$E[Y] = \sum_{i=1}^{10} E[Y_i] = 10 \cdot \frac{1}{19} = \frac{10}{19} \approx 0.526.$$

To compute the variance of Y, note that

$$Var(Y_i) = \frac{1}{19} \cdot \frac{18}{19} = \frac{18}{361}.$$

We also need the covariance of Y_i and Y when $i < j$. First

$$E[Y_i Y_j] = \mathbf{P}\{Y_i = 1, Y_j = 1\} = \frac{1}{19} \cdot \frac{1}{17} = \frac{1}{323}$$

(Once the first couple is chosen, we must choose another couple out of 9 remaining couples.) So, the covariance is

$$Cov(Y_i, Y_j) = E[Y_iY_j] - E[Y_i]E[Y_j] = \frac{1}{323} - \frac{1}{361} = \frac{2}{6137}.$$

Now compute the variance of Y

$$\begin{aligned} Var(Y) &= \sum_{i=1}^{1} 0Var(Y_i) + 2\sum_{i<j} Cov(Y_i, Y_j) \\ &= 10 \cdot \frac{18}{361} + 2 \cdot 45 \cdot \frac{2}{6137}. \\ &= \frac{3240}{6137} \approx 0.528. \end{aligned}$$