MATH 417

Supplemental Midterm 2

Winter, 2008

Name:

Class time:

- **Due** Monday, March 17 at the beginning of class.
- You may use your text, notes or calculator. You **must** work alone.
- Show your work, document your reasons for the solution. No credit will be given if we cannot verify the process that led you to your solution.
- Check your work carefully; you are responsible for your computations. No partial credit will be given for mistakes in computation.

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problem	points	score
1	5	
2	15	
3	10	
Total	30	

1. Find a basis $\mathcal{B}$ for $\mathbb{R}^4$ such that the following matrix represents the identity transformation between the standard basis and $\mathcal{B}$:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 2 & 2 \\ -1 & 1 & -1 & 2 \end{bmatrix}$$

We are looking for vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ for our basis $\mathcal{B}$ satisfying the following four equations:

$$\begin{aligned} \vec{e}_1 &= \vec{v}_1 + \vec{v}_2 - \vec{v}_3 - \vec{v}_4 \\ \vec{e}_2 &= 2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 + \vec{v}_4 \\ \vec{e}_3 &= \vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 - \vec{v}_4 \\ \vec{e}_4 &= \vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 + 2\vec{v}_4 \end{aligned}$$

We can write this as the following matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 2 & 2 \\ -1 & 1 & -1 & 2 \end{bmatrix}$$

This equation can be solved by computing the matrix inverse

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 2 & 2 \\ -1 & 1 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

so the basis $\mathcal{B}$ consists of

$$\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix},$$

2. Let $\mathcal{B}$ be a basis for $\mathcal{R}^3$ consisting of the following vectors

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\2\\2\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by

$$T(\vec{v}_1) = \begin{bmatrix}1\\3\\0\end{bmatrix}, \quad T(\vec{v}_2) = \begin{bmatrix}2\\1\\1\end{bmatrix}, \quad T(\vec{v}_3) = \begin{bmatrix}0\\4\\5\end{bmatrix}$$

(a) Find the matrix for T relative to the standard basis.

(b) Find the matrix for T relative to $\mathcal{B}$.

(c) Find the matrix for T relative to the following basis, given in coordinates for $\mathcal{B}$:

$$[\vec{u}_1]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [\vec{u}_2]_{\mathcal{B}} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad [\vec{u}_3]_{\mathcal{B}} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

(a). We are looking for a matrix A which satisfies the following matrix equation:

$$A \begin{bmatrix}1 & 1 & 1\\1 & 2 & 2\\1 & 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 2 & 0\\3 & 1 & 4\\0 & 1 & 5\end{bmatrix}$$

Solving for A

$$A = \begin{bmatrix}0 & 3 & -2\\5 & -5 & 3\\-1 & -3 & 4\end{bmatrix}$$

A is the matrix for T in standard coordinates.

(b). We are looking for a matrix B satisfying the following matrix equation:

$$B = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 2\\1 & 2 & 3\end{bmatrix}^{-1} A \begin{bmatrix}1 & 1 & 1\\1 & 2 & 2\\1 & 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 2\\1 & 2 & 3\end{bmatrix}^{-1} \begin{bmatrix}1 & 2 & 0\\3 & 1 & 4\\0 & 1 & 5\end{bmatrix}$$

Solving for B

$$B = \begin{bmatrix}-1 & 3 & -4\\5 & -1 & 3\\-3 & 0 & 1\end{bmatrix}$$

B is the matrix for T in $\mathcal{B}$ coordinates.

(c). We are told the matrix R which translates $\mathcal{C}$ coordinates into $\mathcal{B}$ coordinates:

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We are looking for a matrix C satisfying the following matrix equation:

$$C = R^{-1}BR = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & -2 \\ 5 & -5 & 3 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving for C

$$C = \begin{bmatrix} -6 & -2 & -9 \\ 8 & 7 & 9 \\ -3 & -3 & -2 \end{bmatrix}$$

C is the matrix for T in $\mathcal{C}$ coordinates.

You can also translate from $\mathcal{C}$ to standard coordinates: the first matrix translates from $\mathcal{C}$ to standard coordinates, the second translates from standard coordinates back to $\mathcal{C}$:

$$Q = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then,

$$C = Q^{-1}AQ = \begin{bmatrix} 3 & -3 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -4 \\ 5 & -1 & 3 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{bmatrix}$$

Solving for C

$$C = \begin{bmatrix} -6 & -2 & -9 \\ 8 & 7 & 9 \\ -3 & -3 & -2 \end{bmatrix}$$

3. Find orthonormal bases for the following subspace and for its orthogonal complement:

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \ : \ x - y - z + w = 0 \text{ and } x + z = 0 \right\}$$

V is the set of solutions to the following pair of equations

$$\begin{aligned} x - y - z + w &= 0 \\ x + z &= 0 \end{aligned}$$

or equivalently, the kernel to the following matrix

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

So, we can solve for V

$$V = \left\{ \begin{bmatrix} -z \\ w - 2z \\ z \\ w \end{bmatrix} \ : \ z, w \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

We can compute an orthonormal basis for $V^{\perp}$: it is the kernel of the matrix:

$$\begin{bmatrix} -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

So, we can solve for $V^{\perp}$

$$V^{\perp} = \left\{ \begin{bmatrix} z + 2w \\ -w \\ z \\ w \end{bmatrix} \ : \ z, w \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

We get orthonormal bases for V and $V^{\perp}$ by Gram-Schmidt:

$$V : \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad V^{\perp} : \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$