MATH 417
Midterm 2
Winter, 2008

Name:

Class time:

- You have **50 minutes** to complete your work.
- Show all work and make it clear what your answers are.
- Books, notes, calculators and computers are not permitted on this exam.

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problem	points	score
1	15	
2	15	
3	20	
4	5	
5	5	
Total	60	

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\-2\end{bmatrix}\right) = \begin{bmatrix}5\\2\end{bmatrix}$$

(a) Find coordinates for $\vec{e}_1$ and $\vec{e}_2$ with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix} \right\}$.

(b) Find the matrix for T relative to the standard basis.

(c) Find the matrix for T relative to $\mathcal{B}$.

Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\-2\end{bmatrix}$. These equalities give $\vec{v}_1$ and $\vec{v}_2$ in *standard coordinates*, so we can convert coordinates for $\mathcal{B}$ to standard coordinates by the matrix:

$$S = \begin{bmatrix}1 & 1\\1 & -2\end{bmatrix}$$

Then S^{-1} is the matrix that goes from standard coordinates to $\mathcal{B}$ coordinates:

$$S^{-1} = \frac{1}{3}\begin{bmatrix}2 & 1\\1 & -1\end{bmatrix}$$

(a). We can now compute coordinates for the standard basis vectors in $\mathcal{B}$:

$$[\vec{e}_1]_{\mathcal{B}} = S^{-1}\vec{e}_1 = \frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} \qquad [\vec{e}_2]_{\mathcal{B}} = S^{-1}\vec{e}_1 = \frac{1}{3}\begin{bmatrix}1\\-1\end{bmatrix}$$

(b). Compute $T(\vec{e}_1)$ and $T(\vec{e}_2)$:

$$\begin{aligned} T(\vec{e}_1) &= T(\frac{2}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2) \\ &= \frac{2}{3}\begin{bmatrix}-1\\2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}5\\2\end{bmatrix} \\ &= \begin{bmatrix}1\\2\end{bmatrix} \end{aligned}$$

$$\begin{aligned} T(\vec{e}_2) &= T(\frac{1}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2) \\ &= \frac{1}{3}\begin{bmatrix}-1\\2\end{bmatrix} - \frac{1}{3}\begin{bmatrix}5\\2\end{bmatrix} \\ &= \begin{bmatrix}-2\\0\end{bmatrix} \end{aligned}$$

So, the matrix for T in standard coordinates is

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$$

(c). The matrix for T is $\mathcal{B}$ coordinates can be computed by $S^{-1}AS$:

$$\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & 1 \end{bmatrix}$$

This says that

$$T(\vec{v}_1) = -\vec{v}_2 \quad T(\vec{v}_2) = 4\vec{v}_1 + \vec{v}_2$$

which you can confirm.

2. Consider the following matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

(a) Find an orthonormal basis for the image of A.

(b) Find an orthonormal basis for the orthogonal complement of the image of A, $(\operatorname{im} A)^{\perp}$.

(c) Compute the projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ on the image of A.

(a). The first two columns are orthogonal, so we need only normalize them:

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

The orthogonal projection of the third column ($\vec{v}_3$) onto the space spanned by $\vec{u}_1, \vec{u}_2$ is

$$(\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_1) = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

So, the final vector in the orthonormal basis is (before normalizing)

$$\begin{bmatrix} 0 \\ 2 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Thus, our orthonormal basis is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{10}} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

(b). $(\operatorname{im} A)^\perp = \ker(A^T)$, or the kernel of

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

This matrix in reduced-row echelon form is

$$\begin{bmatrix} -1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

so that a basis for the kernel is $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$. The orthonormal basis for $(\operatorname{im} A)^\perp$ is

$$\frac{2}{\sqrt{10}} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

(c). The easiest way to do this is to compute the orthogonal projection directly. Notice that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is orthogonal to the first two vectors in the orthonormal basis from (a). So, its projection is determined solely by the third vector:

$$\frac{1}{10}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \frac{3}{5} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

3. Let S be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$.

(a) Find the standard matrix of the orthogonal projection onto S.

(b) Find the orthogonal projection of the vector $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$ onto S.

(c) Find an orthonormal basis for $S^{\perp}$.

(d) Find the orthogonal projection of the vector $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$ onto $S^{\perp}$.

An orthonormal basis for S is

$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$$

(a). The matrix for the orthogonal projection is

$$\begin{bmatrix}0 & \frac{1}{\sqrt{3}}\\0 & \frac{1}{\sqrt{3}}\\1 & 0\\0 & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}0 & 0 & 1 & 0\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ 0 & 0 & 1 & 0\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix}$$

(b). We can use the matrix computed in (a) to get the orthogonal projection onto S:

$$\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ 0 & 0 & 1 & 0\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ -1\\ \frac{1}{3}\end{bmatrix}$$

(c). $S^{\perp}$ can be computed by the kernel of the following matrix (the transpose of the matrix whose columns are the basis vectors for S):

$$\begin{bmatrix}0 & 0 & 1 & 0\\1 & 1 & 0 & 1\end{bmatrix}$$

so,

$$S^\perp = \left\{ \begin{bmatrix} -x-y \\ x \\ 0 \\ y \end{bmatrix} \quad x, y \in \mathbb{R} \right\}$$

An orthonormal basis for $S^\perp$ is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(d). We can compute the orthogonal projection onto $S^\perp$ directly from (b):

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -1 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ -\frac{1}{3} \end{bmatrix}$$

4. Let $\vec{u}_1, \vec{u}_2$, and $\vec{u}_3$ be an orthonormal basis in $\mathbb{R}^3$. Find a and b for which the following vectors form an orthonormal basis for $\mathbb{R}^3$:

$$\begin{aligned} \vec{v}_1 &= \frac{2}{3}\vec{u}_1 + a\vec{u}_2 + \frac{2}{3}\vec{u}_3 \\ \vec{v}_2 &= -\frac{2}{3}\vec{u}_1 + \frac{2}{3}\vec{u}_2 + \frac{1}{3}\vec{u}_3 \\ \vec{v}_3 &= \frac{1}{3}\vec{u}_1 + \frac{2}{3}\vec{u}_2 + b\vec{u}_3 \end{aligned}$$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be orthonormal, we must have the following equations hold

$$\begin{aligned} 0 = \vec{v}_1 \cdot \vec{v}_2 &= -\frac{4}{9} + \frac{2a}{3} + \frac{2}{9} \\ 0 = \vec{v}_3 \cdot \vec{v}_2 &= -\frac{2}{9} + \frac{4}{9} + \frac{b}{3} \end{aligned}$$

Simplifying

$$\begin{aligned} 0 &= -\frac{2}{9} + \frac{2a}{3} \\ 0 &= \frac{2}{9} + \frac{b}{3} \end{aligned}$$

So, $a = \frac{1}{3}$ and $b = -\frac{2}{3}$.

5. Let A and B be invertible $n \times n$ matrices. Show that $AB + I_n$ is similar to $BA + I_n$.

We want an invertible matrix S such that

$$\begin{aligned} S^{-1}\big(AB + I_n\big) &= S^{-1}\big(AB\big)S + S^{-1}\big(I_n\big)S \\ &= S^{-1}\big(AB\big)S + I_n \\ &= BA + I_n \end{aligned}$$

So, the problem reduces to finding an invertible matrix S with

$$S^{-1}\big(AB\big)S = BA$$

Let $S = A$, then

$$A^{-1}\big(AB\big)A = \big(A^{-1}A\big)BA = BA.$$