MATH 417
Midterm 1
Winter, 2008

Name:

Class time:

- You have **50 minutes** to complete your work.
- Show all work and make it clear what your answers are.
- Books, notes, calculators and computers are not permitted on this exam.

problem	points	score
1	15	
2	10	
3	15	
4	10	
5	5	
6	5	
Total	60	

1. Let E be the following system of equations, where k is a real number:

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 4z &= 6 \\ 3x + 4y + (k^2+2)z &= 9 + k \end{aligned}$$

(a) For which values of k is E inconsistent? (7 pts)

(b) Find all solutions of E for each value k for which E is consistent. (8 pts)

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 2 & 2 & 4 & 6 \\ 3 & 4 & k^2+2 & k+9 \end{array}\right] \xrightarrow{\text{Gauss-Jordan}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & k^2-4 & k-2 \end{array}\right]$$

If $k \neq -2, 2$, then we can divide by k^2-4

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{k+2} \end{array}\right] \xrightarrow{-2\,\mathrm{III}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{k}{k+2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{k+2} \end{array}\right].$$

The system has a unique solution $\left(\frac{k}{k+2}, 2, \frac{1}{k+2}\right)$.

If $k = -2$, then $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -4 \end{array}\right]$ the system is inconsistent.

If $k = 2$, then $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$. The system has infinitely many solutions $(1-2z, 2, z)$.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 7 & 7 & 0 \\ 2 & 1 & 9 & 10 & 2 \\ 2 & 0 & 4 & 6 & 2 \end{bmatrix} \text{ which reduces to } \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find a basis for the image of A. (5pts)

(b) Find a basis for the kernel of A. (5pts)

The image has as a basis the column vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{v}_5 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

To find the kernel, we need to find all $\vec{x}$ such that $B\vec{x} = 0$, where $B = \operatorname{rref}(A)$

$$\left| \begin{array}{l} x_1 + 2x_3 + 3x_4 = 0 \\ x_2 + 5x_3 + 4x_4 = 0 \\ x_5 = 0 \end{array} \right| \Rightarrow \left| \begin{array}{l} x_1 = -2x_3 - 3x_4 \\ x_2 = -5x_3 - 4x_4 \\ x_5 = 0 \end{array} \right|$$

$$\ker(A) = \left\{ \vec{x} = \begin{bmatrix} -2x_3 - 3x_4 \\ -5x_3 - 4x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} : x_3, x_4 \text{ in } \mathbb{R} \right\} \Rightarrow$$

$$\vec{w}_1 = \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{w}_2 = \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis of the $\ker(A)$

3. Consider the following transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + ky \\ kx + y + kz \\ z \end{bmatrix}$$

where k is a real number.

(a) Show T is a linear transformation. (4 pts)

(b) For what real numbers k is T invertible. (4 pts)

(c) Find a matrix for T. (3 pts)

(d) Determine the inverse of T when $k = 2$. (4 pts)

(a)

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+k(y_1+y_2) \\ k(x_1+x_2)+y_1+y_2+k(z_1+z_2) \\ z_1+z_2 \end{bmatrix} =$$

$$= \begin{bmatrix} x_1+ky_1 \\ kx_1+y_1+kz_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2+ky_2 \\ kx_2+y_2+kz_2 \\ z_2 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

(c) $A_T = \begin{bmatrix} 1 & k & 0 \\ k & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A_T \to \begin{bmatrix} 1 & k & 0 \\ 0 & 1-k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, T is Invertible If $k \neq \pm 1$

(d) $A_T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$A_T^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$

4. Let $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ be three linearly independent vectors from $\mathbb{R}^n$. Determine whether the three vectors $\vec{v}_1$, $\vec{v}_1 + \vec{v}_2$ and $\vec{v}_1 + \vec{v}_2 + \vec{v}_3$ are linearly dependent or independent. Justify your answer.

Consider a linear relation

$$c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}, \text{ or}$$

$$(c_1 + c_2 + c_3)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly Independent, we have

$$\left|\begin{array}{r} c_1 + c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \\ c_3 = 0 \end{array}\right| \Longrightarrow \left|\begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{array}\right|$$

5. Suppose A and B are $n \times n$ matrices and $ABA = I_n$. Show that B is invertible and find its inverse.

Since A, BA are $n \times n$ matrices s.t

$$A(BA) = I_n$$

by fact 2.4.9, we have A and BA are ~~is~~ invertible and

$$BA = A^{-1}.$$

Now, multiply by A^{-1} from the right, we get

$$(BA)A^{-1} = A^{-2}$$

$$B(AA^{-1}) = A^{-2}$$

$$B \cdot I_n = A^{-2}$$

$$B = A^{-2}$$

6. Show that there can be no 5×3 matrix A and 3×4 matrix B with $\ker(AB) = \{\vec{0}\}$.

First, ker(AB) contains ker(B).

Indeed, If $\vec{x}$ in ker(B), then $B\vec{x} = \vec{0}$

and $(AB)(\vec{x}) = A(B\vec{x}) = A(\vec{0}) = \vec{0}$.

Second, since $\text{rank}(B) \leq 3 < 4$ the system

$B\vec{x} = \vec{0}$ has infinitely many solutions. Hence,

$\ker(B) \neq \{\vec{0}\}$.